- The search time of BDM and BOM is $\mathcal{O}(n(\log_\sigma m)/m)$, which is optimal on average. (BNDM is optimal only when $m \leq w$.)

- MP and KMP are optimal in the worst case.

- There are also algorithms that are optimal in both cases. They are based on similar techniques, but they will not be described them here.

Karp–Rabin

The Karp–Rabin algorithm uses a hash function $H : \Sigma^* \to [0..q) \subset \mathbb{N}$ for strings. It computes $H(P)$ and $H(T[j..j+m))$ for all $j \in [0..n-m]$.

- If $H(P) \neq H(T[j..j+m))$, then we must have $P \neq T[j..j+m)$.
- If $H(P) = H(T[j..j+m)$, the algorithm compares P and $T[j..j+m)$ in brute force manner. If $P \neq T[j..j+m)$, this is a collision.

A good hash function has two important properties:

- Collisions are rare.
- Given $H(a\alpha)$, a and b, where $a, b \in \Sigma$ and $\alpha \in \Sigma^*$, we can quickly compute $H(\alpha b)$. This is a called rolling or sliding window hash function.

The latter property is essential for fast computation of $H(T[j..j+m))$ for all j.

The hash function used by Karp–Rabin is

$$H(c_0c_1c_2\ldots c_{m-1}) = (c_0r^{m-1} + c_1r^{m-2} + \cdots + c_{m-2}r + c_{m-1}) \bmod q$$

This is a rolling hash function:

$$H(\alpha) = (H(a\alpha) - ar^{m-1}) \bmod q$$
$$H(\alpha b) = (H(\alpha) \cdot r + b) \bmod q$$

which follows from the rules of modulo arithmetic:

$$(x+y) \bmod q = ((x \bmod q) + (y \bmod q)) \bmod q$$
$$(xy) \bmod q = ((x \bmod q)(y \bmod q)) \bmod q$$

The parameters q and r have to be chosen with some care to ensure that collisions are rare.

- The original choice is $r = \sigma$ and q is a large prime.
- Another possibility is $q = 2^w$, where w is the machine word size, and r is a small prime ($r = 37$ has been suggested). This is faster in practice, because it avoids slow modulo operations.
- The hash function can be randomized by choosing q or r randomly. Furthermore, we can change q or r whenever a collision happens.

Algorithm 1.17: Karp-Rabin

Input: text $T = T[0 \ldots n)$, pattern $P = P[0 \ldots m)$
Output: position of the first occurrence of P in T

Choose q and r; $s \leftarrow r^{m-1} \bmod q$
$hp \leftarrow 0; ht \leftarrow 0$
for $i \leftarrow 0$ **to** $m - 1$ **do** $hp \leftarrow (hp \cdot r + P[i]) \bmod q$ // $hp = H(P)$
for $j \leftarrow 0$ **to** $m - 1$ **do** $ht \leftarrow (ht \cdot r + T[j]) \bmod q$
for $j \leftarrow 0$ to $n - m - 1$ **do**
    if $hp = ht$ **then if** $P = T[j \ldots j + m)$ **then** return j
    $ht \leftarrow ((ht - T[j] \cdot s) \cdot r + T[j + m]) \bmod q$
if $hp = ht$ **then if** $P = T[j \ldots j + m)$ **then** return j
return n

On an integer alphabet:

- The worst case time complexity is $\mathcal{O}(mn)$.
- The average case time complexity is $\mathcal{O}(m + n)$.

Karp–Rabin is not competitive in practice, but hashing can be a useful technique in other contexts.

2. Approximate String Matching

Often in applications we want to search a text for something that is similar to the pattern but not necessarily exactly the same.

To formalize this problem, we have to specify what does “similar” mean. This can be done by defining a similarity or a distance measure.

A natural and popular distance measure for strings is the edit distance, also known as the Levenshtein distance.

Edit distance

The edit distance $ed(A, B)$ of two strings A and B is the minimum number of edit operations needed to change A into B. The allowed edit operations are:

S Substitution of a single character with another character.

I Insertion of a single character.

D Deletion of a single character.

Example 2.1: Let $A =$ `Lewensteinn` and $B =$ `Levenshtein`. Then $ed(A, B) = 3$.

The set of edit operations can be described with an edit sequence or with an alignment:

```
NNSNNNINNNND
Lewens-teinn
Levenshtein-
```

In the edit sequence, N means No edit.

There are many variations and extension of the edit distance, for example:

- Hamming distance allows only the subtitution operation.
- Damerau–Levenshtein distance adds an edit operation:

  T Transposition swaps two adjacent characters.

- With weighted edit distance, each operation has a cost or weight, which can be other than one.
- Allow insertions and deletions (indels) of factors at a cost that is lower than the sum of character indels.

We will focus on the basic Levenshtein distance.

Computing Edit Distance

Given two strings $A[1..m]$ and $B[1..n]$, define the values d_{ij} with the recurrence:

$$
\begin{aligned}
d_{00} &= 0, \\
d_{i0} &= i, \;\; 1 \le i \le m, \\
d_{0j} &= j, \;\; 1 \le j \le n, \;\; \text{and} \\
d_{ij} &= \min \begin{cases} d_{i-1,j-1} + \delta(A[i], B[j]) \\ \quad d_{i-1,j} + 1 \\ \quad d_{i,j-1} + 1 \end{cases} \qquad 1 \le i \le m, 1 \le j \le n,
\end{aligned}
$$

where

$$
\delta(A[i], B[j]) = \begin{cases} 1 & \text{if } A[i] \ne B[j] \\ 0 & \text{if } A[i] = B[j] \end{cases}
$$

Theorem 2.2: $d_{ij} = ed(A[1..i], B[1..j])$ for all $0 \le i \le m$, $0 \le j \le n$. In particular, $d_{mn} = ed(A, B)$.

Example 2.3: $A =$ `ballad`, $B =$ `handball`

d		h	a	n	d	b	a	l	l
	0	1	2	3	4	5	6	7	8
b	1	1	2	3	4	4	5	6	7
a	2	2	1	2	3	4	4	5	6
l	3	3	2	2	3	4	5	4	5
l	4	4	3	3	3	4	5	5	4
a	5	5	4	4	4	4	4	5	5
d	6	6	5	5	4	5	5	5	6

$ed(A, B) = d_{mn} = d_{6,8} = 6$.

Proof of Theorem 2.2. We use induction with respect to $i + j$. For brevity, write $A_i = A[1..i]$ and $B_j = B[1..j]$.

Basis:

$$
\begin{aligned}
d_{00} &= 0 = ed(\epsilon, \epsilon) \\
d_{i0} &= i = ed(A_i, \epsilon) \quad (i \text{ deletions}) \\
d_{0j} &= j = ed(\epsilon, B_j) \quad (j \text{ insertions})
\end{aligned}
$$

Induction step: We show that the claim holds for d_{ij}, $1 \le i \le m, 1 \le j \le n$. By induction assumption, $d_{pq} = ed(A_p, B_q)$ when $p + q < i + j$.

The value $ed(A_i, B_j)$ is based on an optimal edit sequence. We have three cases depending on what the last edit operation is:

N or S: $ed(A_i, B_j) = ed(A_{i-1}, B_{j-1}) + \delta(A[i], B[j]) = d_{i-1,j-1} + \delta(A[i], B[j])$.

I: $ed(A_i, B_j) = ed(A_i, B_{j-1}) + 1 = d_{i,j-1} + 1$.

D: $ed(A_i, B_j) = ed(A_{i-1}, B_j) + 1 = d_{i-1,j} + 1$.

Since the edit sequence is optimal, the correct value is the minimum of the three cases, which agrees with the definition of d_{ij}. □

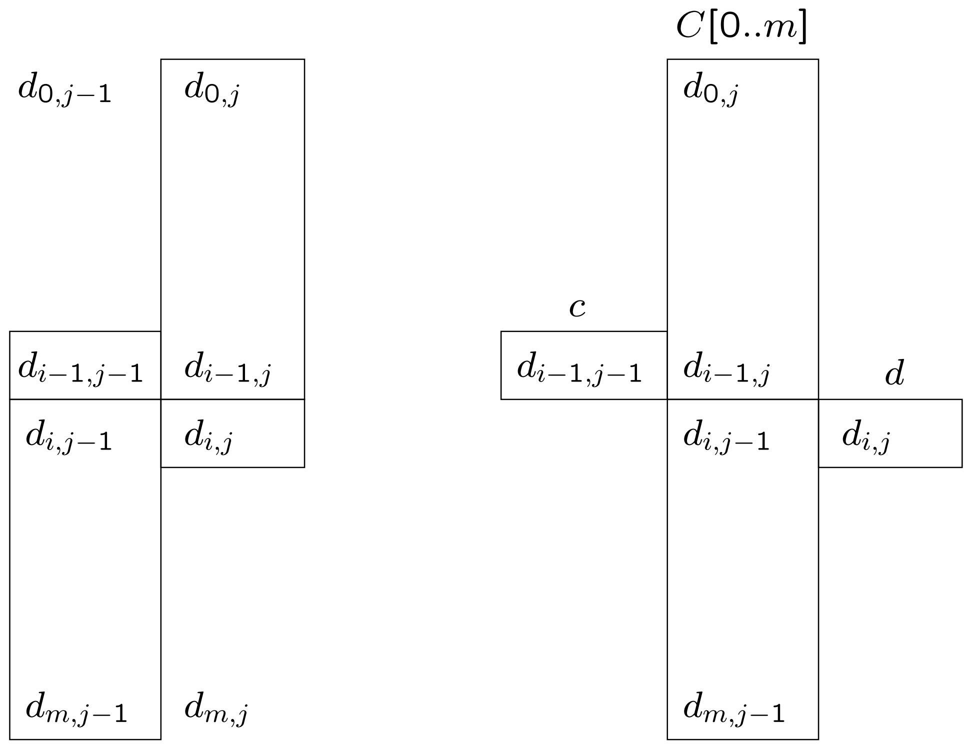
The recurrence gives directly a dynamic programming algorithm for computing the edit distance.

Algorithm 2.4: Edit distance
Input: strings $A[1..m]$ and $B[1..n]$
Output: $ed(A, B)$
(1) for $i \leftarrow 0$ to m do $d_{i0} \leftarrow i$
(2) for $j \leftarrow 1$ to n do $d_{0j} \leftarrow j$
(3) for $j \leftarrow 1$ to n do
(4)   for $i \leftarrow 1$ to m do
(5) $d_{ij} \leftarrow \min\{d_{i-1,j-1} + \delta(A[i], B[j]), d_{i-1,j} + 1, d_{i,j-1} + 1\}$
(6) return d_{mn}

The time and space complexity is $\mathcal{O}(mn)$.

The space complexity can be reduced by noticing that each column of the matrix (d_{ij}) depends only on the previous column. We do not need to store older columns.

A more careful look reveals that, when computing d_{ij}, we only need to store the bottom part of column $j-1$ and the already computed top part of column j. We store these in an array $C[0..m]$ and variables c and d as shown below:



Algorithm 2.5: Edit distance in $\mathcal{O}(m)$ space
Input: strings $A[1..m]$ and $B[1..n]$
Output: $ed(A, B)$

(1) for $i \leftarrow 0$ to m do $C[i] \leftarrow i$
(2) for $j \leftarrow 1$ to n do
(3) $\quad c \leftarrow C[0]$; $C[0] \leftarrow j$
(4) $\quad$ for $i \leftarrow 1$ to m do
(5) $\quad\quad d \leftarrow \min\{c + \delta(A[i], B[j]), C[i-1] + 1, C[i] + 1\}$
(6) $\quad\quad c \leftarrow C[i]$
(7) $\quad\quad C[i] \leftarrow d$
(8) return $C[m]$

- Note that because $ed(A, B) = ed(B, A)$ (exercise), we can assume that $m \leq n$.

It is also possible to find optimal edit sequences and alignments from the matrix d_{ij}.

An edit graph is a directed graph, where the nodes are the cells of the edit distance matrix, and the edges are as follows:

- If $A[i] = B[j]$ and $d_{ij} = d_{i-1,j-1}$, there is an edge $(i-1, j-1) \to (i, j)$ labelled with N.
- If $A[i] \neq B[j]$ and $d_{ij} = d_{i-1,j-1} + 1$, there is an edge $(i-1, j-1) \to (i, j)$ labelled with S.
- If $d_{ij} = d_{i,j-1} + 1$, there is an edge $(i, j-1) \to (i, j)$ labelled with I.
- If $d_{ij} = d_{i-1,j} + 1$, there is an edge $(i-1, j) \to (i, j)$ labelled with D.

Any path from $(0, 0)$ to (m, n) is labelled with an optimal edit sequence.

Example 2.6: $A =$ `ballad`, $B =$ `handball`

d		h		a		n		d		b		a		l		l	
	0	⇒	1	⇒	2	⇒	3	⇒	4	→	5	→	6	→	7	→	8
b	↓	⇘		↘		↘		↘		⇘							
	1		1	→	2	→	3	→	4		4	→	5	→	6	→	7
a	↓	↘	↓	⇘								⇘					
	2		2		1	⇒	2	→	3	→	4		4	→	5	→	6
l	↓	↘	↓		↓	⇘		⇘		↘		↘	↓	⇘		↘	
	3		3		2		2	⇒	3	→	4	→	5		4	→	5
l	↓	↘	↓		↓	↘	↓	⇘		⇘		↘		↘	↓	⇘	
	4		4		3		3		3	⇒	4	→	5		5		4
a	↓	↘	↓	↘	↓	↘	↓	↘	↓	↘		⇘					⇓
	5		5		4		4		4		4		4	⇒	5		5
d	↓	↘	↓		↓	↘	↓	↘		↘	↓	↘	↓	⇘		⇘	⇓
	6		6		5		5		4	→	5		5		5	⇒	6

There are 7 paths from $(0,0)$ to $(6,8)$ corresponding to, for example, the following edit sequences and alignments:

```
IIIINNNNDD    SNISSNIS    SNSSINSI
----ballad    ba-lla-d    ball-ad-
handball--    handball    handball
```