Potenssi- ja murtolausekkeiden derivointi

Tehtävät:

1. Derivoi.

   a) $x^5 + x^4 + x^2$ b) $4x^{10} - 2x^8 + 3x^6$ c) $3x^2 - 5x - 97$

2. Derivoi.

   a) $(2x+1)^2$ b) $(3x-4)(x+x^2)$

   c) $\dfrac{5x^8 - 4x^5 + x}{x}$ d) $\dfrac{6x^5 + 5x^4 + x^3 - x^2}{2x^2}$

3. Derivoi.

   a) $\dfrac{x}{x+3}$ b) $\dfrac{x^2}{x+5}$ c) $\dfrac{3x^2+2}{x+1}$

4. Derivoi.

   a) $\dfrac{x(x^2+1)}{3x+1}$ b) $\dfrac{(x+5)^2}{x-4}$ c) $\dfrac{(2x-1)(2x+1)}{(x+2)^2}$

5. Derivoi.

   a) $x^{-3} + x^{-1}$ b) $\dfrac{x^3 + 3x - 4}{x^2}$ c) $\dfrac{(2x-1)(3x+2)}{x}$

6. Määritä funktion $f(x) = \dfrac{3x^2 - x}{2x+1} - \dfrac{7x}{8}$ derivaatan nollakohdat.

7. Millä muuttujan x arvoilla funktion $f(x) = \dfrac{x^2 + x + 2}{x}$ derivaatta on negatiivinen?

8. Derivoi ja ilmoita tulos potenssimuodossa.

   a) $x^{2/3}$ b) $x^{3/2}$ c) $x^{-1/2}$

   d) $x^2 \cdot x^{-1/3}$ e) $(1-x^2)^{3/2}$ f) $(1+x^{4/5})^{5/4}$

9. Derivoi ja ilmoita tulos juurimuodossa.

   a) $\sqrt[5]{x}$ b) $\sqrt[5]{x^3}$ c) $x\sqrt{x}$

   d) $\dfrac{1}{\sqrt{x}}$ e) $\dfrac{1}{\sqrt[3]{1-x}}$ f) $x\sqrt[3]{6x+1}$

10. Derivoi.

a) $(2x+2)\sqrt{x}$ b) $x+\sqrt[3]{x}$ c) $\dfrac{x+1}{\sqrt[4]{x}}$

Ratkaisut:

1.

a) $D(x^5 + x^4 + x^2) = 5x^4 + 4x^3 + 2x$

b) $D(4x^{10} - 2x^8 + 3x^6) = 4 \cdot 10x^9 - 2 \cdot 8x^7 + 3 \cdot 6x^5 = 40x^9 - 16x^7 + 18x^5$

c) $D(3x^2 - 5x - 97) = 3 \cdot 2x - 5 = 6x - 5$

2.

a) $D\big((2x+1)^2\big) = D(4x^2 + 4x + 1) = 8x + 4$

b) $D\big((3x-4)(x+x^2)\big) = D(3x^2 + 3x^3 - 4x - 4x^2) = D(3x^3 - x^2 - 4x)$
$= 9x^2 - 2x - 4$

c) $D\left(\dfrac{5x^8 - 4x^5 + x}{x}\right) = D(5x^7 - 4x^4 + 1) = 35x^6 - 16x^3$

d) $D\left(\dfrac{6x^5 + 5x^4 + x^3 - x^2}{2x^2}\right) = D\left(3x^3 + \dfrac{5}{2}x^2 + \dfrac{1}{2}x - 1\right) = 9x^2 + 5x + \dfrac{1}{2}$

3.

a) $D\left(\dfrac{x}{x+3}\right) = \dfrac{Dx \cdot (x+3) - x \cdot D(x+3)}{(x+3)^2} = \dfrac{1 \cdot (x+3) - x \cdot 1}{(x+3)^2}$
$= \dfrac{3}{(x+3)^2}$

b) $D\left(\dfrac{x^2}{x+5}\right) = \dfrac{D\ x^2 \cdot (x+5) - x^2 \cdot D(x+5)}{(x+5)^2} = \dfrac{2x \cdot (x+5) - x^2 \cdot 1}{(x+5)^2}$
$= \dfrac{2x^2 + 10x - x^2}{(x+5)^2} = \dfrac{x^2 + 10x}{(x+5)^2}$

c) $D\left(\dfrac{3x^2 + 2}{x+1}\right) = \dfrac{D(3x^2+2) \cdot (x+1) - (3x^2+2) \cdot D(x+1)}{(x+1)^2}$
$= \dfrac{6x \cdot (x+1) - (3x^2+2) \cdot 1}{(x+1)^2} = \dfrac{6x^2 + 6x - 3x^2 - 2}{(x+1)^2} = \dfrac{3x^2 + 6x - 2}{(x+1)^2}$

4.

a) $$D\left(\frac{x(x^2+1)}{3x+1}\right) = D\left(\frac{x^3+x}{3x+1}\right)$$
$$= \frac{D(x^3+x)\cdot(3x+1) - (x^3+x)\cdot D(3x+1)}{(3x+1)^2}$$
$$= \frac{(3x^2+1)(3x+1) - (x^3+x)\cdot 3}{(3x+1)^2} = \frac{9x^3+3x^2+3x+1-3x^3-3x}{(3x+1)^2}$$
$$= \frac{6x^3+3x^2+1}{(3x+1)^2}$$

b) $$D\left(\frac{(x+5)^2}{x-4}\right) = D\left(\frac{x^2+10x+25}{x-4}\right)$$
$$= \frac{D(x^2+10x+25)\cdot(x-4) - (x^2+10x+25)\cdot D(x-4)}{(x-4)^2}$$
$$= \frac{(2x+10)(x-4) - (x^2+10x+25)\cdot 1}{(x-4)^2}$$
$$= \frac{2x^2-8x+10x-40-x^2-10x-25}{(x-4)^2} = \frac{x^2-8x-65}{(x-4)^2}$$

c) $$D\left(\frac{(2x-1)(2x+1)}{(x+2)^2}\right) = D\left(\frac{4x^2-1}{(x+2)^2}\right)$$
$$= \frac{D(4x^2-1)\cdot(x+2)^2 - (4x^2-1)\cdot D(x+2)^2}{((x+2)^2)^2}$$
$$= \frac{8x(x+2)^2 - (4x^2-1)\cdot(2(x+2)\cdot 1)}{(x+2)^4}$$
$$= \frac{(x+2)(8x(x+2) - (4x^2-1)\cdot 2)}{(x+2)^4} = \frac{8x(x+2) - (4x^2-1)\cdot 2}{(x+2)^3}$$
$$= \frac{8x^2+16x-8x^2+2}{(x+2)^3} = \frac{16x+2}{(x+2)^3}$$

5.

a) $D(x^{-3} + x^{-1}) = -3x^{-4} + (-1)x^{-2} = -3x^{-4} - x^{-2}$

b) $D\left(\dfrac{x^3 + 3x - 4}{x^2}\right) = D((x^3 + 3x - 4)(x^{-2})) = D(x + 3x^{-1} - 4x^{-2})$
$= 1 + 3 \cdot (-1)x^{-2} - 4 \cdot (-2)x^{-3} = 1 - 3x^{-2} + 8x^{-3}$

c) $D\left(\dfrac{(2x-1)(3x+2)}{x}\right) = D((6x^2 + 4x - 3x - 2) \cdot x^{-1})$
$= D((6x^2 + x - 2) \cdot x^{-1}) = D(6x + 1 - 2x^{-1}) = 6 + 0 - 2 \cdot (-1)x^{-2}$
$= 6 + 2x^{-2}$

6. Lavennetaan ensin lausekkeet samannimisiksi ja vähennetään:

$$f(x) = \frac{3x^2 - x}{2x+1} - \frac{7x}{8} = \frac{8(3x^2 - x)}{8(2x+1)} - \frac{(2x+1)7x}{(2x+1)8}$$
$$= \frac{24x^2 - 8x}{16x+8} - \frac{14x^2 + 7x}{16x+8} = \frac{24x^2 - 8x - 14x^2 - 7x}{16x+8} = \frac{10x^2 - 15x}{16x+8}.$$

Sitten derivoidaan:

$$f'(x) = D\left(\frac{10x^2 - 15x}{16x+8}\right)$$
$$= \frac{D(10x^2 - 15x) \cdot (16x+8) - (10x^2 - 15x) \cdot D(16x+8)}{(16x+8)^2}$$
$$= \frac{(20x - 15) \cdot (16x+8) - (10x^2 - 15x) \cdot 16}{(16x+8)^2}$$
$$= \frac{320x^2 + 160x - 240x - 120 - 160x^2 + 240x}{(16x+8)^2} = \frac{160x^2 + 160x - 120}{(16x+8)^2}$$
$$= \frac{40(4x^2 + 4x - 3)}{(16x+8)^2}.$$

Murtolausekkeen nollakohdat ovat samat kuin osoittajan nollakohdat:

$$40(4x^2 + 4x - 3) = 0 \iff 4x^2 + 4x - 3 = 0$$
$$\iff x = \frac{-4 \pm \sqrt{16 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4} = \frac{-4 \pm \sqrt{64}}{8} = \frac{-4 \pm 8}{8}.$$

Derivaatan nollakohdat ovat siis $x_1 = 1/2$ ja $x_2 = -3/2$.

7. Derivoidaan funktio:

$$f'(x) = D\left(\frac{x^2+x+2}{x}\right) = D((x^2+x+2)\cdot x^{-1}) = D(x+1+2x^{-1})$$
$$= 1+0+2\cdot(-1)x^{-2} = 1-2x^{-2} = 1-\frac{2}{x^2}.$$

Derivaatta ei ole määritelty, kun $x = 0$ (kuten ei itse funktiokaan). Ratkaistaan derivaatan nollakohdat:

$$1-\frac{2}{x^2} = 0 \iff 1 = \frac{2}{x^2} \iff x^2 = 2 \iff x = \pm\sqrt{2}.$$

Koska derivaatta on määrittelyjoukossaan jatkuva, se voi vaihtaa merkkiä vain nollakohdissa tai niissä kohdissa, jotka eivät kuulu määrittelyjoukkoon. Täytyy siis tutkia merkki väleillä $]-\infty, -\sqrt{2}[$, $]-\sqrt{2}, 0[$, $]0, \sqrt{2}[$ ja $]\sqrt{2}, \infty[$. Lasketaan derivaatan arvoja näillä väleillä:

$$f'(-2) = 1-\frac{2}{(-2)^2} = 1-\frac{2}{4} = \frac{1}{2} > 0,$$
$$f'(-1) = 1-\frac{2}{(-1)^2} = 1-\frac{2}{1} = 1-2 = -1 < 0,$$
$$f'(1) = 1-\frac{2}{1^2} = 1-\frac{2}{1} = 1-2 = -1 < 0,$$
$$f'(2) = 1-\frac{2}{2^2} = 1-\frac{2}{4} = \frac{1}{2} > 0.$$

Derivaatta on siis negatiivinen väleillä $]-\sqrt{2}, 0[$ ja $]0, \sqrt{2}[$.

8.

a) $D\ x^{2/3} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$

b) $D\ x^{3/2} = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}$

c) $D\ x^{-1/2} = -\frac{1}{2}x^{-1/2-1} = -\frac{1}{2}x^{-3/2}$

d) $D(x^2\cdot x^{-1/3}) = D(x^{5/3}) = \frac{5}{3}x^{5/3-1} = \frac{5}{3}x^{2/3}$

e) $D\left((1-x^2)^{3/2}\right) = \frac{3}{2}(1-x^2)^{3/2-1}\cdot D(1-x^2) = \frac{3}{2}(1-x^2)^{1/2}\cdot(-2x)$
$= -3x(1-x^2)^{1/2}$

f) $D\left((1+x^{4/5})^{5/4}\right) = \frac{5}{4}(1+x^{4/5})^{5/4-1}\cdot D(1+x^{4/5})$
$= \frac{5}{4}(1+x^{4/5})^{1/4}\cdot\frac{4}{5}x^{-1/5} = x^{-1/5}(1+x^{4/5})^{1/4}$

9.

a) $D\ \sqrt[5]{x} = D\ x^{1/5} = \frac{1}{5}x^{1/5-1} = \frac{1}{5}x^{-4/5} = \frac{1}{5\sqrt[5]{x^4}}$

b) $D\ \sqrt[5]{x^3} = D\ x^{3/5} = \frac{3}{5}x^{3/5-1} = \frac{3}{5}x^{-2/5} = \frac{3}{5\sqrt[5]{x^2}}$

c) $D(x\sqrt{x}) = D(x \cdot x^{1/2}) = D(x^{3/2}) = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$

d) $D\left(\frac{1}{\sqrt{x}}\right) = D\ x^{-1/2} = -\frac{1}{2}x^{-1/2-1} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2\sqrt{x^3}}$

e) $D\left(\frac{1}{\sqrt[3]{1-x}}\right) = D\ (1-x)^{-1/3} = -\frac{1}{3}(1-x)^{-1/3-1} \cdot D(1-x)$
$= -\frac{1}{3}(1-x)^{-4/3} \cdot (-1) = \frac{1}{3\sqrt[3]{(1-x)^4}}$

f) $D\big(x\sqrt[3]{6x+1}\big) = D\big(\sqrt[3]{x^3}\sqrt[3]{6x+1}\big) = D\big(\sqrt[3]{x^3(6x+1)}\big) = D\big(\sqrt[3]{6x^4+x^3}\big)$
$= D\ (6x^4+x^3)^{1/3} = \frac{1}{3}(6x^4+x^3)^{1/3-1} \cdot D(6x^4+x^3)$
$= \frac{1}{3}(6x^4+x^3)^{-2/3} \cdot (24x^3+3x^2) = (6x^4+x^3)^{-2/3} \cdot (8x^3+x^2)$
$= ((6x+1)x^3)^{-2/3} \cdot (8x+1)x^2 = (6x+1)^{-2/3}(x^3)^{-2/3} \cdot (8x+1)x^2$
$= (6x+1)^{-2/3}x^{-2} \cdot (8x+1)x^2 = (6x+1)^{-2/3} \cdot (8x+1) = \frac{8x+1}{\sqrt[3]{(6x+1)^2}}$

10. Derivoi.

a) $D\big((2x+2)\sqrt{x}\big) = D\big((2x+2) \cdot x^{1/2}\big) = D(2x^{3/2}+2x^{1/2})$
$= 2 \cdot \frac{3}{2}x^{3/2-1} + 2 \cdot \frac{1}{2}x^{1/2-1} = 3x^{1/2} + x^{-1/2} = 3\sqrt{x} + \frac{1}{\sqrt{x}}$

b) $D\big(x + \sqrt[3]{x}\big) = D\big(x + x^{1/3}\big) = 1 + \frac{1}{3}x^{1/3-1} = 1 + \frac{1}{3}x^{-2/3} = 1 + \frac{1}{3\sqrt[3]{x^2}}$

c) $D\left(\frac{x+1}{\sqrt[4]{x}}\right) = D\big((x+1) \cdot x^{-1/4}\big) = D\big(x^{3/4} + x^{-1/4}\big)$
$= \frac{3}{4}x^{3/4-1} - \frac{1}{4}x^{-1/4-1} = \frac{3}{4}x^{-1/4} - \frac{1}{4}x^{-5/4} = \frac{3}{4\sqrt[4]{x}} - \frac{1}{4(\sqrt[4]{x})^5}$
$= \frac{3(\sqrt[4]{x})^4}{4(\sqrt[4]{x})^5} - \frac{1}{4(\sqrt[4]{x^3})^5} = \frac{3(\sqrt[4]{x})^4 - 1}{4(\sqrt[4]{x})^5} = \frac{3x-1}{4(\sqrt[4]{x})^5}$