Y100 (Mathematics I)
Exercise 5 (2 pages)
3.12.–6.12.2007

1. Show that the given function is a solution to the differential equation in the following cases.

| | | |
|---|---|---|
| a) | Equation: $y' = 2xy$ | Function: $y(x) = e^{(x^2)}$ |
| b) | Equation: $x'''x''x' = 0$ | Function: $x(t) = 5t^2 + 3t - 1$ |
| c) | Equation: $y'' - 2y' + 2y = 0$ | Function: $y(t) = e^t \sin t$ |

2. The time evolution of a bacterial population is studied using the model of exponential growth. Days are used as the unit of time. In the beginning (day 0) the mass of the population is approximately 1 mg. In a previous study, the reproduction rate used in the model had been estimated to be 1,5.
   a) Calculate a prediction for the amount of bacteria to be observed after one week.
   b) After one week the mass of the population was observed to be 20 g (i.e. 20000 mg). What seems to be wrong with the observation? Does it fix the situation, if the fact that the culture plate can only support 50 g of bacterial mass is taken account, and the model of logistic growth is used instead?

3. Identify those equations that are separable. Transform them into their separated form ($g(y)y' = h(x)$), and solve any one of them.

   a) $\sqrt{y} \cdot y' = 3x + 2$, b) $y' = y + 1$, c) $y' = y + x$,
   d) $y'x = y' + y$, e) $2xy' = 2xy - 1$.

4. Solve the following initial value problems. (Remember to take account the possible particular solutions that are found before the separation.)

   a) $y'' = x, \quad y(0) = 0, \quad y'(0) = 1,$

   b) $\dfrac{y'}{x} = \dfrac{3x}{y}, \quad y(1) = 2, \quad (\text{assume } x > 0),$

   b) $y' = (2x + 1)y, \quad y(10) = 0.$

5. Some substance flows into a container with no outlets. Let $T(t)$ denote the amount of substance in the container at time $t$. In the beginning (when $t = 0$) there was 5 litres of substance in the container. The rate of the incoming flow $s(t)$ is proportional to the amount of substance in the container, so that $s(t) = k(t)T(t)$, where the factor of proportionality $k$

follows the function $k(t) = e^{-t}$. Form a differential equation describing the situation, and draw a suitable flow diagram. Solve the resulting equation by separating. As it is known that $\lim_{t\to\infty} e^{-t} = 0$, what can you say about the amount of substance that will eventually gather into the container?

6. The decay rate of radioactive matter is at each moment directly proportional to the amount of substance still left to decay. Denote the amount of substance $m(t)$. In the beginning, there was 100 kg of the substance, and after a year, only 50 kg was left (this means also that the *half life* of the substance is one year).
   a) Form a differential equation describing the situation, and find out its solution.
   b) Using the initial condition $m(0) = 100$, solve the value of the integration constant. Then use the given condition $m(1) = 50$ to solve the value of the proportionality constant. This value should be approximately $-0{,}693$.
   c) Interpret the situation as a flow model, where substance is removed from the container, and draw a flow diagram of the situation.