Y100 (Mathematics 1)
Exercise 3 (2 pages)
19.–22.11.2007

1. A farmer's wife is going to use the eggs that her husband's hens have produced, to bake a cake whose base has the form of a square. The sides and the top of the cake will be covered with cream. The wife has enough cream to cover a surface of 1200 cm^2 in area. How thick should the cake be made, so that there would be enough cream, but at the same time the volume of the cake would be as large as possible?

   Hint: Choose as the variable x the width of the base of the cake. Then the area of one side will be $x \cdot$ "height of cake". Deduce from the amount of cream available, how much cream can be used for each side, and from this, how the thickness of the cake depends on x. The function for the volume is of course $x^2 \cdot$ "thickness of cake", which should become $V(x) = -\frac{1}{4}x^3 + 300x$.

2. Some substance runs into a container at speed a, and the same substance flows out from the container at speed b. The speed of the incoming flow is at each moment of time inversely proportional to the amount of the substance inside the container (that is, $a(t) = k_1 \cdot \frac{1}{\text{“amount of substance”}}$); on the other hand, the speed of the outgoing flow is at each moment directly proportional to the amount of the substance inside the container. Moreover, the proportionality constant of the incoming flow k_1 is two times as great as the proportionality constant of the outgoing flow.

   Draw a flux diagram of the situation, and write an equation describing the situation. In the equation, there should appear the (unknown) function describing the amount of the substance and also the derivative of that function.

3. Let $f(x) = x^2$. Compute the upper and lower sums over the interval $[0, 2]$ using a partition into four subintervals (so the length of each subinterval is $1/2$). Draw also the graph of the function and the rectangles that approximate the graph from above and from below. Then calculate the exact integral $\int_0^2 f(x)\,dx$, and compare your results.

subinterval	max. value	area of upper rect.	min. value	a. of lower rect.
$[0, 1/2]$				
$[1/2, 1]$				
$[1, 3/2]$				
$[3/2, 2]$				
sum	—		—	

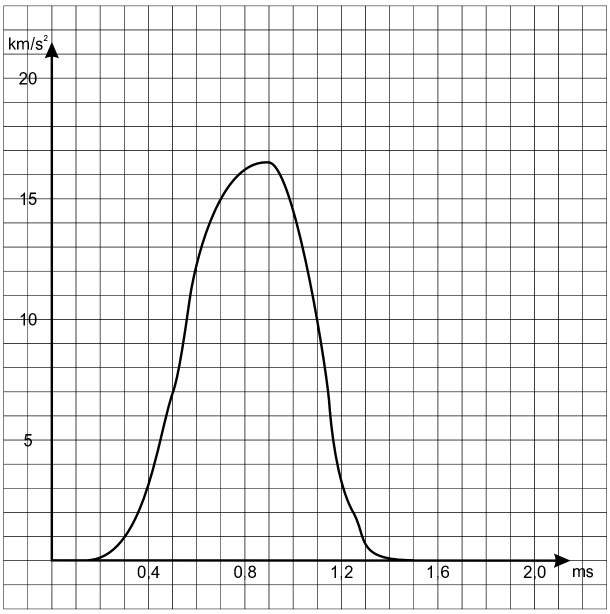
4. Compute the following integrals.

a) $\displaystyle\int_{-1}^{1} 6x^2 - 2x + 1\,dx$, b) $\displaystyle\int_{1}^{2} x^2 - \frac{1}{x^2}\,dx$, c) $\displaystyle\int_{0}^{1} t + \sqrt{t}\,dt$,

d) $\displaystyle\int_{0}^{5} f(x)\,dx$, where $f(x) = \begin{cases} 3x^2, & \text{when } x \geq 1, \\ x/4, & \text{when } x < 1 \end{cases}$.

5. In physics, the force used to move a body is directly proportional to the acceleration the body acquires, the proportionality constant being the mass of the body. The acceleration, on the other hand, describes change in the velocity of the body.

A force sensor was attached to a billiard ball, and the ball was struck. The acceleration of the ball at each instant was calculated from the reading of the force sensor by dividing the force by the ball's mass. In this way, the graph of the acceleration of the ball was formed. Approximate by integrating from the graph below the velocity the ball acquired during the striking. (The area of one square is $0{,}1 \cdot 1 = 0{,}1$, in m/s.)



6. Using the rule for differentiating a composed function, show that the function $F(x) = (x^2 - 1)^8$ is an integral function of the function $f(x) = 16x(x^2 - 1)^7$. Then calculate the following integral:

$$\int_{0}^{1} 16x(x^2 - 1)^7\,dx.$$

Find an integral function for the function $g(x) = 3x(x^2 - 1)^7$.