Algebra II
Department of Mathematics and Statistics
Problem sheet 10
Thu 15.4.2010

1. An algebra is said to be *finitely generated* if it contains a finite subset $X = \{x_1, \ldots, x_n\}$, such that every element of the algebra is a linear combination of finite products of elements in X. Show that every finitely generated unital, associative and commutative R-algebra is isomorphic to a quotient algebra of some polynomial algebra over the coefficient ring R.

2. For each algebra A appearing in Question 5 of Problem sheet 9, find a corresponding ideal of the two-dimensional polynomial algebra $\mathbb{R}[X]$, such that $\mathbb{R}[X]/I \cong A$.

3. a) Show that under the cross product, the space $\mathbb{R}^3$ becomes a Lie algebra.
   b) Show that the matrices

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   form a basis for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. What are the structure constants of $\mathfrak{sl}_2(\mathbb{R})$ with respect to this basis?

4. Let K be a field. Prove the following claims:
   a) If $\operatorname{char}(K) = p > 0$, then p is a prime number.
   b) If $\operatorname{char}(K) = p > 0$, then every subfield of K contains a subfield isomorphic to $\mathbb{F}_p$.
   c) If $\operatorname{char}(K) = 0$, every subfield of K contains a subfield isomorphic to $\mathbb{Q}$.

5. Let K be a field. Assume that $f \in K[X]$ is an irreducible polynomial. Prove that the principal ideal generated by f is a prime ideal.

   Hint: If $gh \in \langle f \rangle$, an appeal to the division algorithm yields $f = gq + r$. Show that $f|(rh)$.

6. Prove that the polynomials $X^2 - X + 1$ and $X^3 + X + 1$ are irreducible in the ring $\mathbb{F}_2[X]$, but not in the ring $\mathbb{F}_3[X]$. Form the corresponding extensions of $\mathbb{F}_2$, and solve the equations $a^2 = 1$ and $b^4 + b^2 = 1$ in both of them.