Algebra II
Department of Mathematics and Statistics
Problem sheet 8
Thu 25.3.2010

1. Show that every free subset of the $\mathbb{Z}$-module $\mathbb{Q}$ contains at most one element, and conclude that $\mathbb{Q}$ is not a free group.

2. Let M and N be R-modules. Suppose that the module M has a basis B, and let $f : B \to N$ be a mapping. Let $\varphi : M \to N$ be an R-linear map, such that $\varphi(b) = f(b)$ for all $b \in B$ (cf. Theorem 8.2). Prove that

   i) The map φ is injective if and only if the image fB is a free subset.

   ii) The map φ is surjective if and only if the image fB generates N.

3. Prove that every commutative group is a quotient of a free commutative group.

   Hint: Choose a generating set X for the group G, and consider the free module $\mathbb{Z}^{(X)}$. Use the Homomorphism Theorem to a suitable linear map $\varphi : \mathbb{Z}^{(X)} \to G$.

4. Let M be a finite commutative group. Describe the module $\mathbb{Q} \otimes_{\mathbb{Z}} M$.

5. Assume that M, N and P are R-modules, and $\varphi : M \to N$ is an isomorphism. Show that the following isomorphisms exist:

   a) $M \otimes P \cong N \otimes P$, where $x \otimes y \mapsto \varphi(x) \otimes y$

   b) $R \otimes M \cong M$, where $a \otimes x \mapsto a.x$

   c) $(M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P)$, missä $(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$.

   In part (b) the ring R is thought of as an R-module.

   Hint: For part (c), construct linear maps $\psi_1 : M \otimes P \to (M \oplus N) \otimes P$, such that $x \otimes z \mapsto (x, 0) \otimes z$, and $\psi_2 : N \otimes P \to (M \oplus N) \otimes P$, such that $y \otimes z \mapsto (0, y) \otimes z$. Then define $\psi : (M \otimes P) \oplus (N \otimes P) \to (M \oplus N) \otimes P$ by $\psi(u, v) = \psi_1(u) + \psi_2(v)$.

6. Consider the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Z}$-modules.

   a) Show that there exists a $\mathbb{Z}$-linear map $\varphi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, for which we have $\varphi(x \otimes y) = xy$.

   b) Show that the map $\psi : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\psi(x) = x \otimes 1$, is surjective and the inverse of φ.