Algebra II
Department of Mathematics and Statistics
Problem sheet 7 (2 pages)
Thu 18.3.2010

1. Let R a commutative ring, and suppose $S^{-1}R$ is the division ring of R with respect to the subset S. Prove the following claims (Theorem 6.11):

   (a) $S^{-1}R$ is a commutative ring, with compositions $a/b \cdot c/d = (ac)/(bc)$ and $a/b + c/d = (ad + bc)/(bd)$.

   (b) The canonical map $\eta : a \mapsto a/1$ is a ring homomorphism.

   (c) If $s \in S$, element $\eta(s) \in S^{-1}R$ has a multiplicative inverse.

   (d) The canonical map η is one-to-one if and only if S does not contain zero divisors.

   (e) $S^{-1}R$ is the zero ring if and only if $0 \in S$.

2. Let R be a ring, and suppose $a \in R$ has no multiplicative inverse. Show that $\langle a \rangle \neq R$.

3. Suppose R is a local ring. Show that the maximal ideal of R consists of precisely those elements of the ring that have no multiplicative inverse.

4. Let $p = X^2 - Y$, and suppose $g \in \mathbb{R}[X, Y]$ is another polynomial. Prove the following claims (cf. Example 6.14):

   (a) The polynomial p is irreducible in the ring $\mathbb{R}[X, Y]$, that is, it cannot be written as a product of non-constant polynomials.

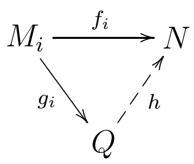
   (b) The polynomial g can be written in the form $g = pq + r$, where $q \in \mathbb{R}[X, Y]$ and $r \in \mathbb{R}[X]$.

   (c) If the polynomials g and p have infinitely many common zeros (x, y), then p divides g.

   Hint: If the polynomial p has non-constant divisors, they must be of the form $aX + bY + c$. The polynomial g can be written in the form $g = \sum_{i=0}^{n} f_i Y^i$, where $f_i \in \mathbb{R}[X]$ for all i. Use induction on n (the highest power of the variable Y).

5. Let G be a finite abelian group. Show that as a $\mathbb{Z}$-module, G is a direct sum of p-groups. (Recall the theorems of group theory.)

6. Let I be an arbitrary index set. Suppose that Q is an R-module, such that for any $i \in I$ there exists an R-module M_i and an R-linear map $g_i : M_i \to Q$. Assume further that the following theorem holds for Q (cf. Theorem 7.4, the universal property of direct sums): if N is another R-module, and the mappings $f_i : M_i \to N$ are R-linear for each i, then there exists a unique R-linear map $h : Q \to N$, such that $h \circ g_i = f_i$ for all i.



Show that Q is isomorphic to the direct product $\bigoplus_i M_i$. In other words, the universal property determines the direct product up to isomorphism.

Hint: Use the theorem in the assumption with Theorem 7.4 to find linear maps $\theta : \bigoplus_i M_i \to Q$ and $h : Q \to \bigoplus_i M_i$. Then use the same theorems for the diagrams shown below to prove that $h \circ \theta = \text{id}$ and $\theta \circ h = \text{id}$.