Algebra II
Department of Mathematics and Statistics
Problem sheet 6
to 4.3.2010

1. Consider the ideal geneated by the polynomials $f = X^4 - 1$ and $g = X^3 + X$ in the polynomial ring $\mathbb{Z}[X]$.
   (a) Find $h \in \mathbb{Z}[X]$, such that $\langle h \rangle = \langle f, g \rangle$.
   (b) Show that in the quotient ring $\mathbb{Z}[X]/\langle f, g \rangle$ there exists an element a, such that $a^2 = -1$.

2. Let R be a ring with an ideal A. Prove the following claims (Theorem 6.6):
   (a) A is a prime ideal if and only if R/A is an integral domain.
   (b) A is maximal if and only if R/A is a field.

3. Assume that A is an ideal in a ring R, and B is a maximal ideal of the quotient ring R/A. Let π denote the canonical surjection $R \to R/A$. Show that $\pi^{-1}B$ is a maximal ideal of the ring R, and A is contained in B. (Corollary 6.10.)

4. Prove the following claims:
   (i) If $f : M \to N$ is a module homomorphism, then $\operatorname{Im} f$ is a submodule of N, and $\operatorname{Ker} f$ is a submodule of M.
   (ii) If (M_i) is a family of submodules of M, their sum $\sum_i M_i$ consists of elements $\sum_i x_i$, where $x_i \in M_i$ and $x_1 = 0$ apart from a finite set of indices.

5. Let R be a ring with an ideal A and a subring S. Prove the following claims:
   (i) Every R/A-module is also an R-module, but not all R-modules are R/A-modules.
   (ii) Every R-module is also an S-module, but not all S-modules are R-modules.

   Hint: Think about $\mathbb{Z}_n$-modules.

6. Let M and N be R-modules. Show that $\operatorname{Hom}_R(M, N)$ is an R-module, and that $\operatorname{Hom}_R(R, M) \cong M$. (The ring R is thought of as a module equipped with its own multiplication.)