Algebra II
Department of Mathematics and Statistics
Problem sheet 4
Thu 18.2.2010

1. Find all inner automorphisms (i.e. those that arise from conjugation) of D_8, the symmetry group of a square. Which group do they form? Show that not all automorphisms of D_8 are inner.

   Hint. Inner automorphisms do not map elements out from their conjugacy classes, and no automorphism can change the order of an element.

2. (a) Let G be a group and $Z(G)$ its centre. Show that if the quotient group $G/Z(G)$ is cyclic, then G is abelian.

   (b) Let p be a prime. Show that every group of order p^2 is abelian.

   Hint. Use the fact that the centre of any p-group is non-trivial.

3. Show that a group of order 80 cannot be simple.

4. Prove Cauchy's Theorem: if a prime p divides the order of a group G, there exists an element $g \in G$, whose order is p.

5. Assume that $m, n \in \mathbb{N}$, and $\gcd(m, n) = 1$. Prove the following claims:

   (a) If $am = bn$ for some $a, b \in \mathbb{N}$, then $m|b$ and $n|a$.

   (b) $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

6. Let G be a group with subgroups H and N. If $G = HN$, $N \trianglelefteq G$ and $H \cap N = \{1\}$, the group G is called a *semidirect product* of its subgroups H and N. Prove the following facts about semidirect products.

   (i) Each $g \in G$ has a unique representation as $g = hn$, where $h \in H$ and $n \in N$.

   (ii) The product $h_1 n_1 \cdot h_2 n_2$, where $h_1, h_2 \in H$ and $n_1, n_2 \in N$, can be written as $h_1 h_2 \cdot n' n_2$, where $n' \in N$ only depends on elements h_2 and n_1.

   (iii) $G/N \cong H$.