Algebra II
Department of Mathematics and Statistics
Problem sheet 2
Thu 4.2.2010

Questions 1 and 2 are related to the quotient structure in Question 3.b) from last week. Let the quotient monoid in question be denoted P (capital ρ). Its elements are the equivalence classes $\overline{0}$, $\overline{1}$, ..., $\overline{r+p-1}$. Consider the difference monoid $E = P \times P/\sim$.

1. Show that the canonical homomorphism $\eta : P \to E$, where $\eta(\overline{a}) = [(\overline{a}, 0)]$, is surjective, and
$$\eta(\overline{a}) = \eta(\overline{b}) \iff a \equiv b \mod p.$$

2. (a) Show that there exists a monoid homomorphism $\overline{f} : P \to \mathbb{Z}_p$, such that $\overline{f}(\overline{n}) = [n]_p$ for all $n \in \mathbb{N}$.
   (b) Show that there is a surjective homomorphism $\overline{g} : E \to \mathbb{Z}_p$, such that $\overline{g}([(\overline{m}, \overline{n})]) = \overline{f}(\overline{m}) - \overline{f}(\overline{n})$ for all $m, n \in \mathbb{N}$.
   (c) Show that $\overline{g}$ is an isomorphism.

3. Assume that the group G acts on the set X on the left. Write Y^X for the set of all maps from X into Y. If $g \in G$ and $\varphi : X \to Y$, define $\varphi^g : X \to Y$ via the formula $\varphi^g(x) = \varphi(gx)$. Show that the formula $\varphi \mapsto \varphi^g$ defines a right action of G in the set of mappings Y^X.

4. Assume that the group G acts on the set X. Prove the following claims.
   (a) The orbits of elements form a partition of the set X.
   (b) Every orbit is a homogeneous (i.e. transitive) G-set.
   (c) The point stabilisers are subgroups of G, not necessarily normal. (Consider e.g. the natural action of S_3 on a set of three elements.)

5. Prove *Cayley's Theorem*: Every group is isomorphic to a permutation group.

   Hint: Consider the action of a group on itself, defined by $g.x = gx$.

6. Suppose p is a prime, and m is a positive integer. Show that if the order of G is p^m, then its center $Z(G)$ is non-trivial.

   Hint: Use the class equation. An element $x \in G$ is contained in the center if and only if $[G : C_G(x)] = 1$.