Algebra II
Department of Mathematics and Statistics
Problem sheet 1
Thu 28.1.2010

1. Let G be a group, and H its subgroup. Show that the cosets of H form a partition of G, and that the following are equivalent:
   (i) $xH = yH$
   (ii) $x \in yH$
   (iii) $x, y \in zH$ for some $z \in G$.

2. Consider the ring of residue classes $\mathbb{Z}_{10} = \mathbb{Z}/10\mathbb{Z}$, having as its elements the residue classes $\overline{n}$ modulo 10, and equipped with the usual addition and multiplication of residue classes. Show that the subset $\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\} \subset \mathbb{Z}_{10}$ is a ring, but not a subring of $\mathbb{Z}_{10}$.

3. In the following structures, find out whether the given equivalence relation (or the one determined by the given partition) and composition law are compatible. If they are, describe the resulting quotient structure. (Check that the given relations really are equivalences, if you are not certain.)
   (a) Partition to squares and non-squares in $(\mathbb{Q}^*, \cdot)$.
   (b) Partition to squares and non-squares in $(\mathbb{Z}_5 \setminus \{\overline{0}\}, \cdot)$.
   (c) The following relation in $(\mathbb{N}, +)$: Let $p, r \in \mathbb{N}$. Define for any $m, n \in \mathbb{N}$, where $m \leq n$,
   $$m \sim n \iff \begin{cases} m = n, & \text{if } m < r, \\ m - n = kp \text{ for some } k \in \mathbb{Z}, & \text{otherwise.} \end{cases}$$

4. Prove the following claims related to the difference monoid example 1.5:
   (a) Relation $\sim$ is an equivalence relation that is compatible with addition.
   (b) Mapping $a \mapsto [(a, 0)]$ in one-to-one if and only if $a + s = b + s$ implies $a = b$ for every $s \in M$.
   (c) The division monoid of $(\mathbb{Z}, \cdot)$ is trivial.

5. Suppose that m divides n. Show that there exists a unique group homomorphism $g : \mathbb{Z}_n \to \mathbb{Z}_m$, such that $g(\overline{1}) = \overline{1}$.

   Hint: Use Theorem 1.14.

6. Prove the following version of Theorem 1.14: Assume that $f : G \to H$ and $p : G \to G'$ are group homomorphisms, and p is surjective. Then there is a homomorphism $\overline{f} : G' \to H$, for which $f = \overline{f} \circ p$, if and only if $\operatorname{Ker} p \subset \operatorname{Ker} f$.