I/O-Efficient Algorithms and Data Structures

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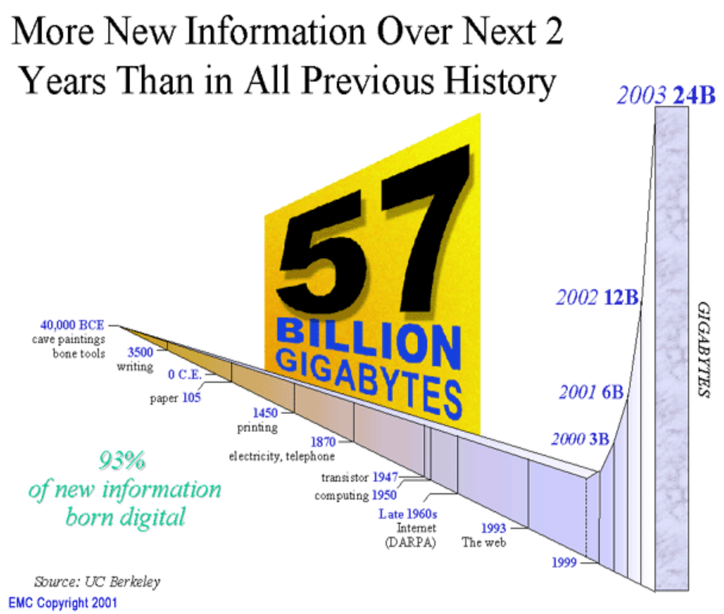
Massive Data

- Pervasive use of computers and sensors
- Increased ability to acquire, store and process data

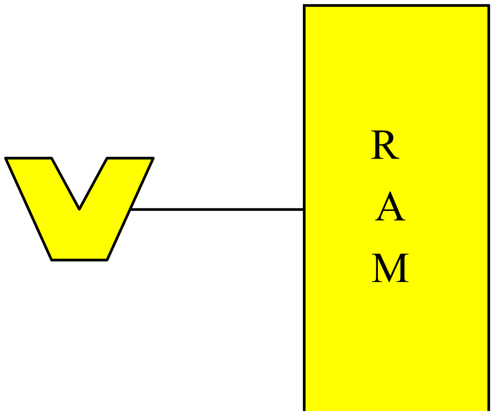
→ Massive data collected everywhere



Examples (2002):

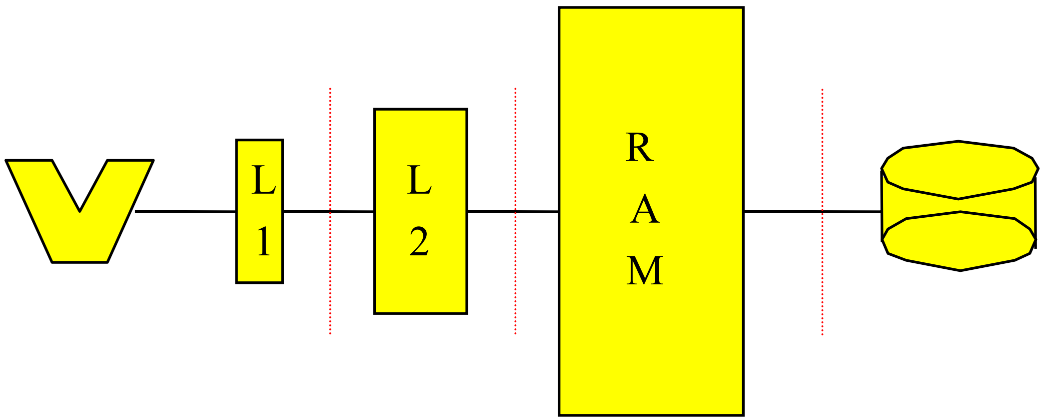
- Phone: AT&T 20TB phone call database, wireless tracking
- Consumer: WalMart 70TB database, buying patterns
- WEB/Network: Google index $8*10^9$ pages, internet routers
- Geography: NASA satellites generate TB each day

Random Access Machine Model



- Standard theoretical model of computation:
  - Infinite memory
  - Uniform access cost
- Simple model crucial for success of computer industry

Hierarchical Memory

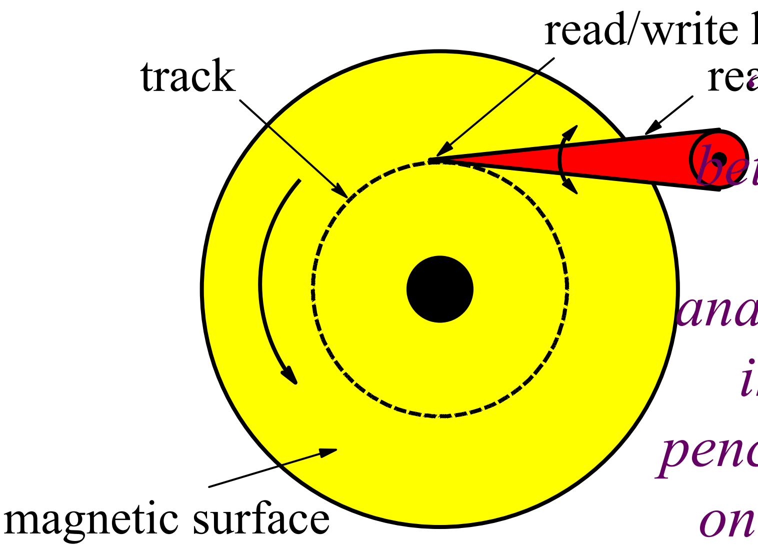


- Modern machines have complicated memory hierarchy
  - Levels get larger and slower further away from CPU
  - Data moved between levels using large blocks

- Bottleneck often transfers between largest memory levels in use

Slow I/O

- Disk access is 10^6 times slower than main memory access

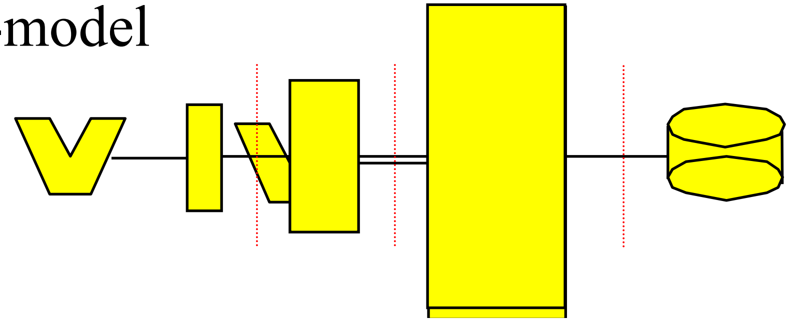


"The difference in speed between modern CPU and disk technologies is analogous to the difference in speed in sharpening a pencil using a sharpener on one's desk or by taking an airplane to the other side of the world and using a sharpener on someone else's desk." (D. Comer)

  - Disk systems try to amortize large access time transferring large contiguous blocks of data (8-16Kbytes)
- Important to store/access data to take advantage of blocks (locality)

Scalability Problems

- Most programs developed in RAM-model
  - Run on large datasets because OS moves blocks as needed



- Moderns OS utilizes sophisticated paging and prefetching strategies
  - But if program makes scattered accesses even good OS cannot take advantage of block access

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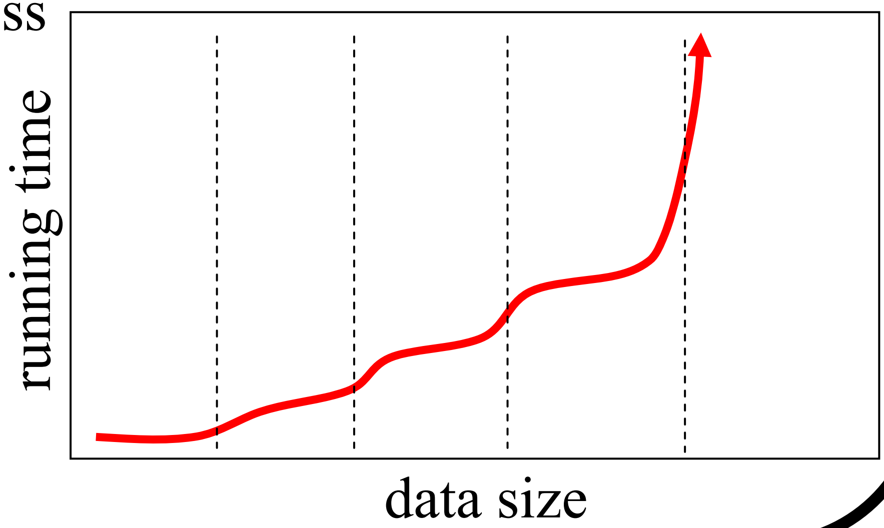
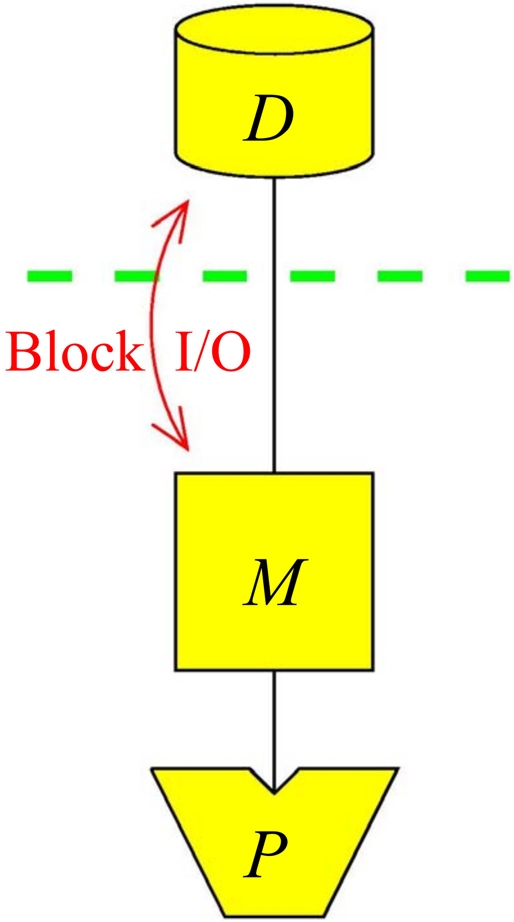
Scalability problems!



External Memory Model



N = # of items in the problem instance

B = # of items per disk block

M = # of items that fit in main memory

T = # of items in output

I/O: Move block between memory and disk

We assume (for convenience) that $M > B^2$

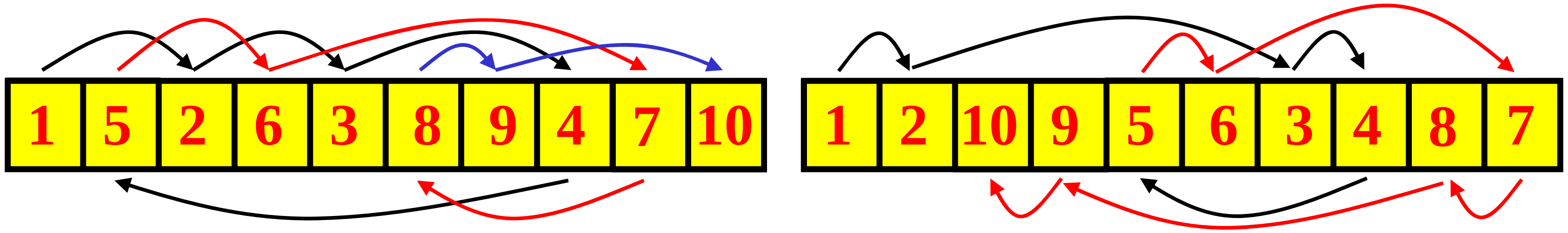
Fundamental Bounds

	Internal	External
• Scanning:	N	$\frac{N}{B}$
• Sorting:	$N \log N$	$\frac{N}{B} \log_{M/B} \frac{N}{B}$
• Permuting	N	$\min\{N, \frac{N}{B} \log_{M/B} \frac{N}{B}\}$
• Searching:	$\log_2 N$	$\log_B N$

- Note:
  - Linear I/O: $O(N/B)$
  - Permuting not linear
  - Permuting and sorting bounds are equal in all practical cases
  - B factor VERY important: $\frac{N}{B} < \frac{N}{B} \log_{M/B} \frac{N}{B} << N$
  - Cannot sort optimally with search tree

Scalability Problems: Block Access Matters

- Example: Traversing linked list (List ranking)
  - Array size N = 10 elements
  - Disk block size B = 2 elements
  - Main memory size M = 4 elements (2 blocks)



Algorithm 1: N=10 I/Os

Algorithm 2: N/B=5 I/Os

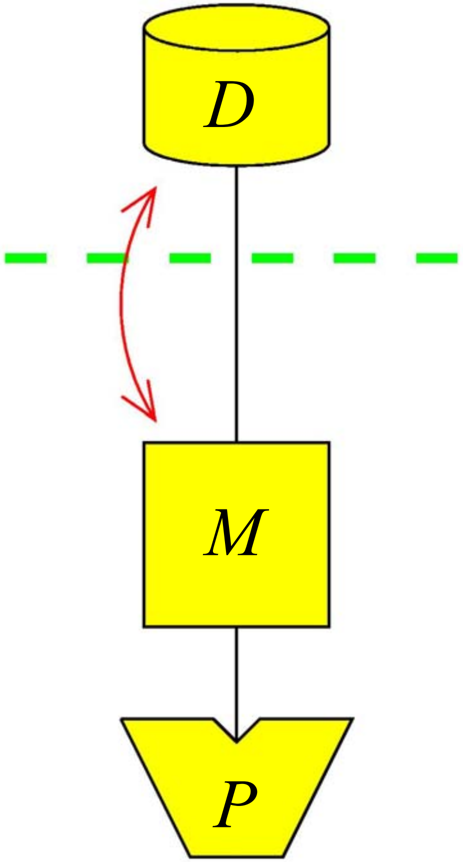
- Difference between N and N/B large since block size is large
  - Example: $N = 256 \times 10^6$, $B = 8000$, *1ms* disk access time

    $\Rightarrow$ ***N*** I/Os take 256×10^3 *sec* = *4266 min* = *71 hr*

    $\Rightarrow$ ***N/B*** I/Os take *256/8 sec* = *32 sec*

Outline

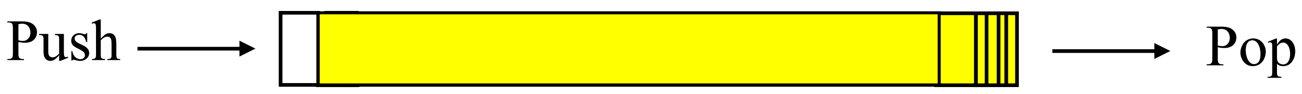
1. Introduction
2. Fundamental algorithms
   a) Sorting
   b) searching
3. Buffered data structures
4. Range searching
5. List ranking



Note: Find references in handouts

Queues and Stacks

- Queue:
  - Maintain push and pop blocks in main memory



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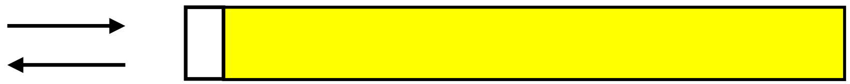
$O(1/B)$ Push/Pop operations

- Stack:
  - Maintain push/pop blocks in main memory



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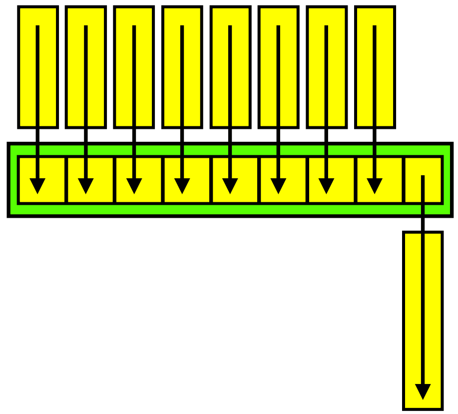
$O(1/B)$ Push/Pop operations

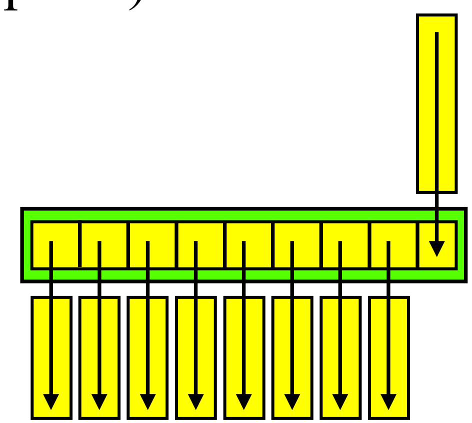
Merging

- $<M/B$ sorted lists (queues) can be merged in $O(N/B)$ I/Os



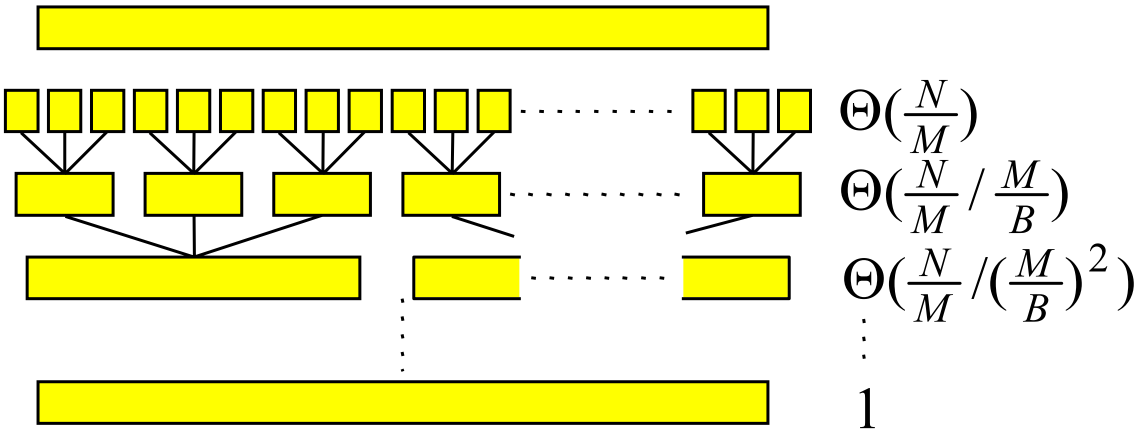
M/B blocks in main memory

- Unsorted list (queue) can be distributed using $<M/B$ split elements in $O(N/B)$ I/Os



Sorting

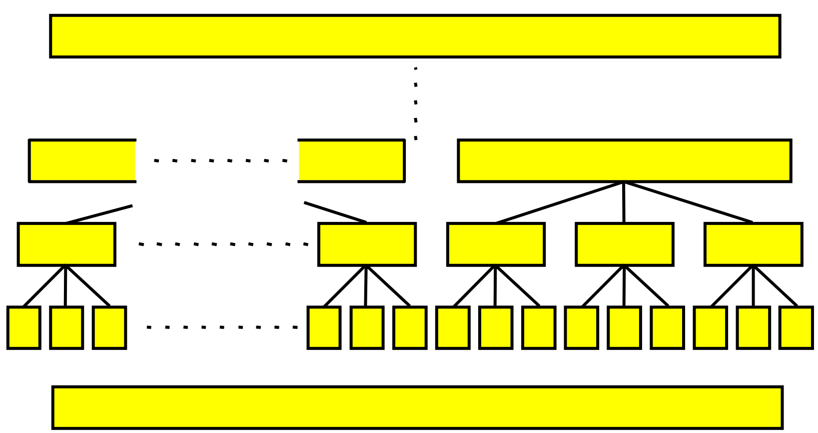
- Merge sort:
  - Create N/M memory sized sorted lists
  - Repeatedly merge lists together $\Theta(M/B)$ at a time



$\Rightarrow O(\log_{M/B} \frac{N}{M})$ phases using $O(N/B)$ I/Os each $\Rightarrow O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/Os

Sorting

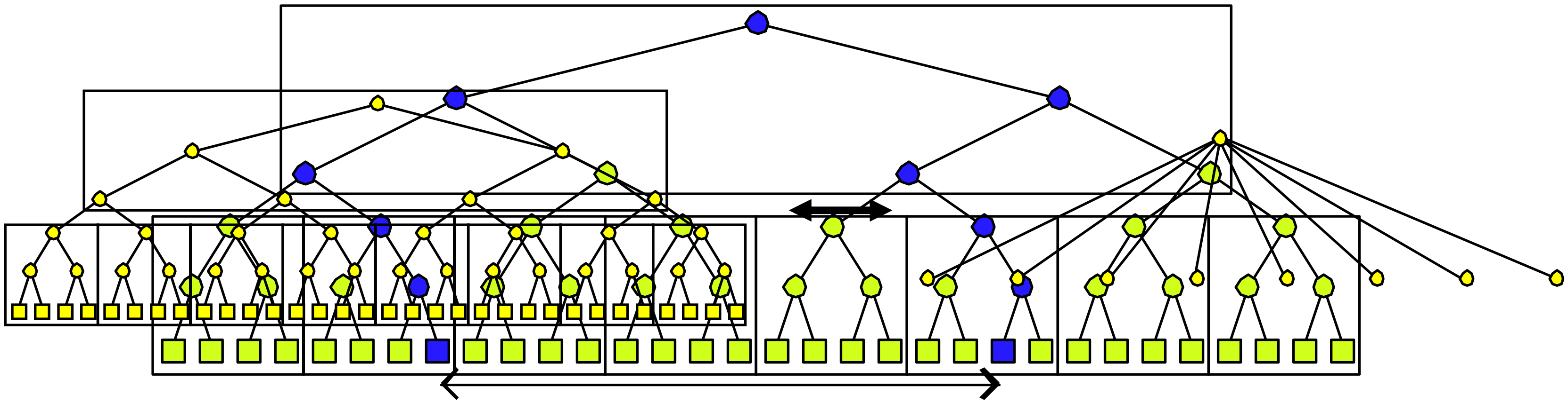
- Distribution sort (multiway quicksort):
  - Compute M/B splitting elements
  - Distribute unsorted list into M/B unsorted lists of equal size
  - Recursively split lists until fit in memory
- We cannot compute M/B splitting elements in $O(N/B)$ I/O
  - But we can compute $\Theta(\sqrt{M/B})$ elements



$\Rightarrow O(\log_{\sqrt{M/B}} \frac{N}{M}) = O(\log_{M/B} \frac{N}{M})$ phases using $O(N/B)$ I/Os each

Searching

- Storing binary trees arbitrarily on disk ⇒ $O(\log N+T)$ query/update

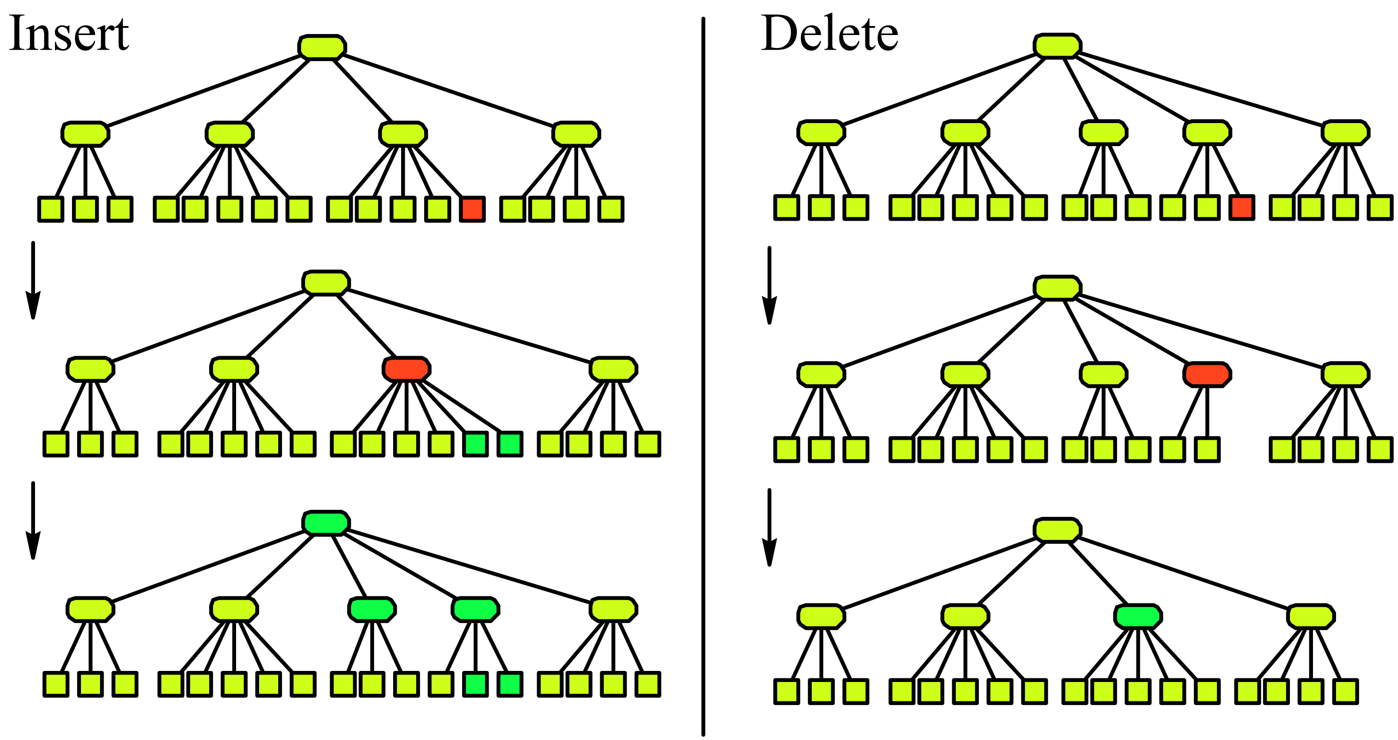


  - blocking B nodes together ⇒ $O(\log_B N+T/B)$
- B-tree
  - All leaves – consisting of $\Theta(B)$ input elements – on same level
  - Internal nodes degree $\Theta(B)$

⇒ $O(N)$ space, $O(\log_B N+T/B)$ range query

Searching: B-tree update

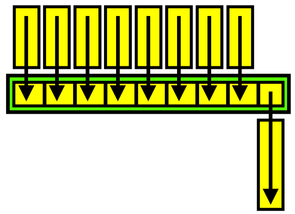
- Blocking hard to maintain using e.g rotations
- Rebalancing using split/fuse (and share):

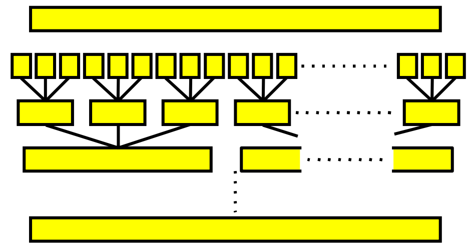


$\Rightarrow O(\log_B N)$ update bound

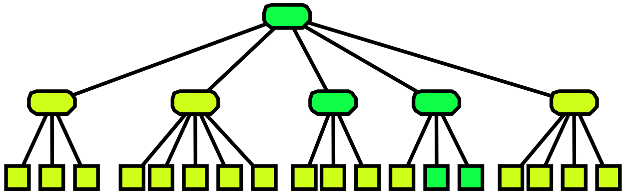
Summary: Fundamental Algorithms

- M/B-way merge/distribution in $O(N/B)$ I/Os $\Rightarrow$
- External merge or distribution sort takes $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os





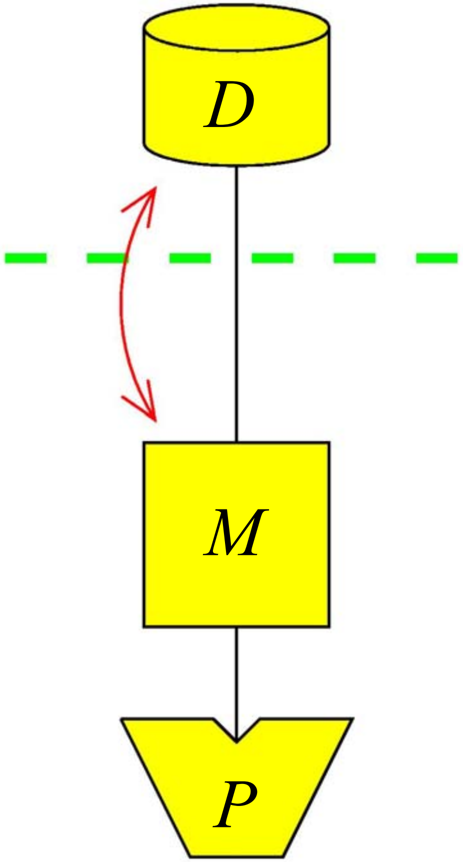
- Fanout $\Theta(B)$ search tree $\Rightarrow$ B-tree
  - $O(\log_B N)$ I/O search/update
  - $O(\log_B N+T/B)$ I/O query



Refs: [A] sec. 1-2, [AV] sec. 1-3, 5

Outline

1. Introduction
2. Fundamental algorithms
3. Buffered data structures
    a) Buffer-tree
    b) Buffered priority queue
4. Range searching
5. List ranking



Buffered Data Structures

- Use of the (on-line) efficient B-tree in external memory algorithms does not lead to efficient algorithms

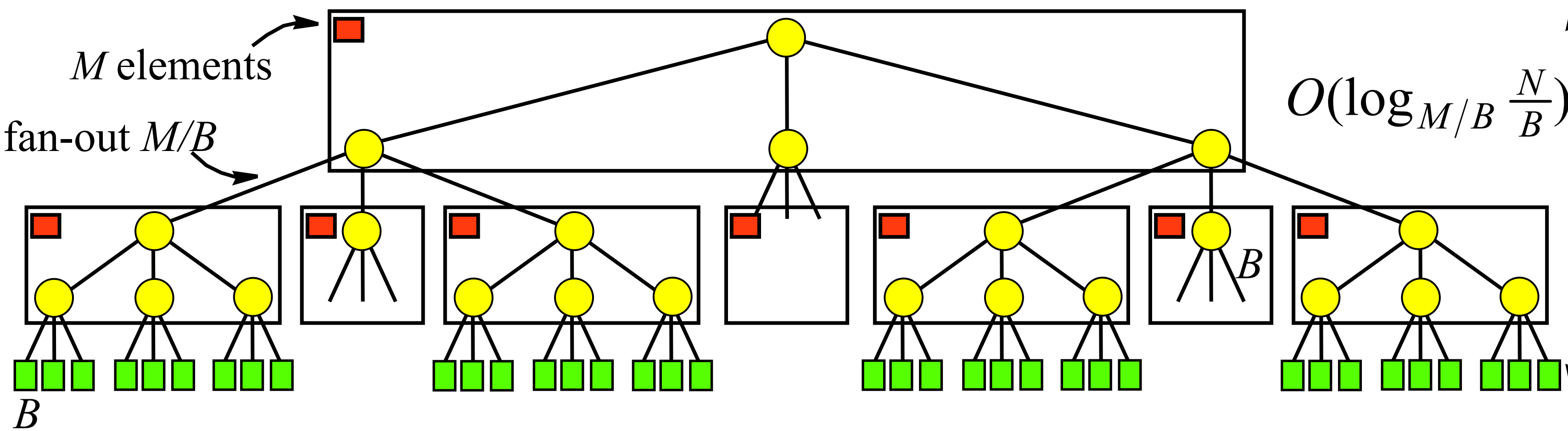
- Example: Sorting using search tree
  - Insert all elements in search tree one-by-one (construct tree)
  - Output in sorted order using in-order traversal

  $\Rightarrow$ Optimal $O(N \log N)$ time in internal memory

  $\Rightarrow$ non-optimal $O(N \log_B N)$ I/Os in external memory

- Need $O(\frac{1}{B} \log_{M/B} \frac{N}{B})$ operations to obtain efficient algorithms
  - $O(N) \cdot O(\frac{1}{B} \log_{M/B} \frac{N}{B}) = O(\frac{N}{B} \log_{M/B} \frac{N}{B})$

Buffer-tree



- Main idea: Logically group nodes together and add buffers
  - Insertions done in a "lazy" way – elements inserted in buffers.
  - When a buffer runs full elements are pushed one level down.
  - Buffer-emptying in $O(M/B)$ I/Os

    $\Rightarrow$ every *block* touched constant number of times on each level

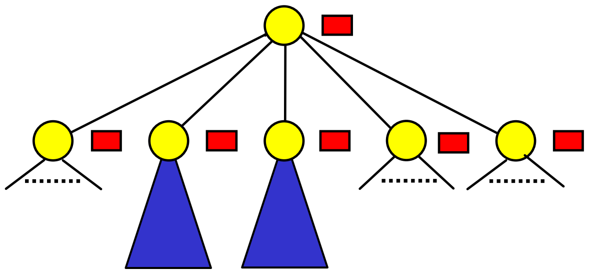
    $\Rightarrow$ inserting N elements (N/B blocks) costs $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os.

Buffer-tree

- Insert (and deletes) on buffer-tree takes $O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ I/Os amortized

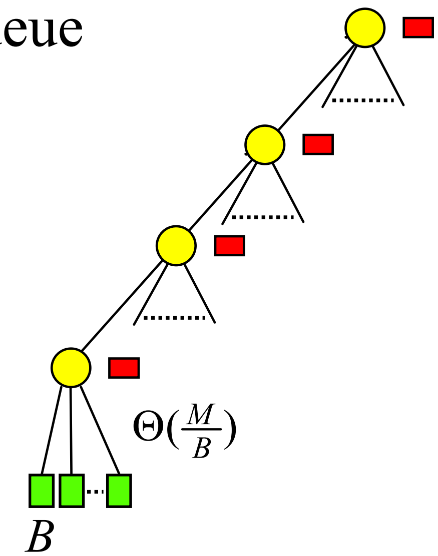
  $\Rightarrow$ Buffer tree can be used in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ sorting algorithm

- One-dim. rangesearch operations can also be supported in $O(\frac{1}{B}\log_{M/B}\frac{N}{B}+\frac{T}{B})$ I/Os amortized
  - Search elements handle lazily like updates
  - All elements in relevant sub-trees reported during buffer-emptying
  - Buffer-emptying in $O(X/B+T'/B)$, where T' is reported elements



Buffered Priority Queue

- Buffer-tree can also be used in external priority queue
- To delete minimal element
  - Empty all buffers on leftmost path
  - Delete M elements in leftmost leaves and keep in memory (Insertions checked against minimal elements)



$\Downarrow$

$O(\frac{M}{B}\log_{M/B}\frac{N}{B})$ I/Os every $O(M)$ delete $\Rightarrow O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ amortized

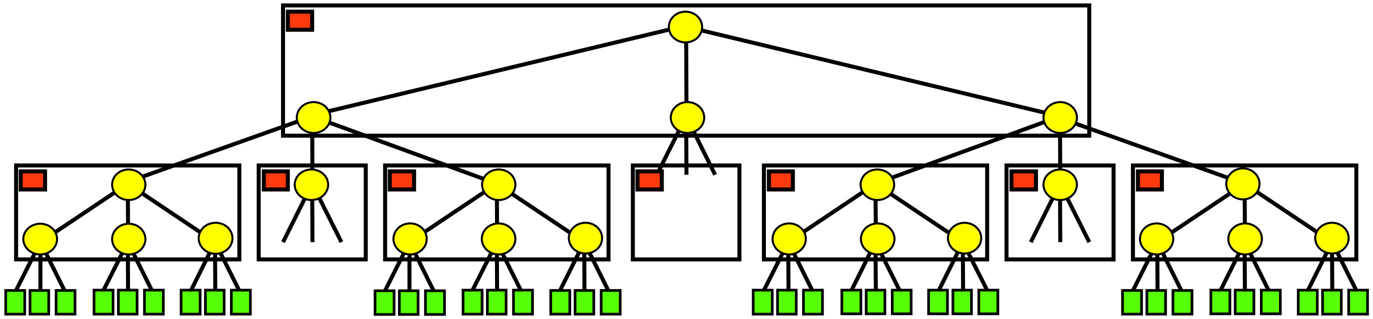
- Buffer technique can also be used on heap and tournament tree

Summary: Buffered Data Structures

- Lazy operations using buffers

  $\Rightarrow O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ I/O amortized operations

- Can for example be used to obtain
  - $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/O B-tree construction algorithm
  - Efficient (on line) priority queue



Refs: [A] sec 5

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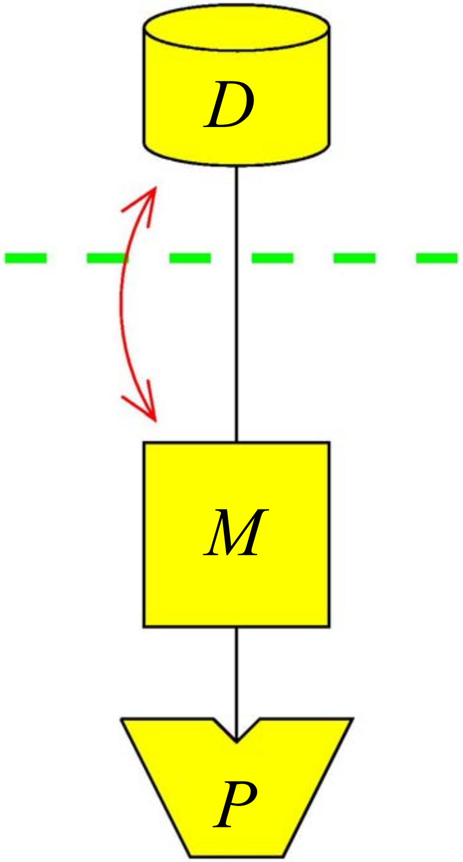
3. Buffered data structures
4. Range searching
5. List ranking



Exercises

1) Design an algorithm for removing duplicates from a multiset.

The output from the algorithm should be the K distinct elements among the N input elements in sorted order.

The algorithm should use $O(\max\{\frac{N}{B}, \frac{N}{B}\log_{M/B}\frac{N}{B} - \sum_{i=1}^{K}\frac{N_i}{B}\log_{M/B}\frac{N_i}{B}\})$ I/Os, where N_i is the number of copies of the i'th element

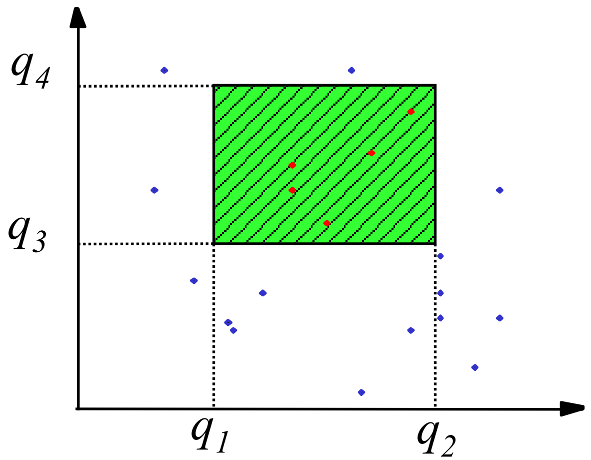
– *Hint:* Modify merge-sort to remove copies as soon as found

2) Design a I/O-efficient version of a heap that supports insert and deletemin operations in $O(\frac{1}{B}\log_{M/B}\frac{N}{B})$ I/Os amortized.

– *Hint/one idea:* Let the heap have fanout *M/B* (rather than *2*) and store *M* minimal elements in each node (rather than one). Buffer *M* inserts in memory before performing them.

External Planar Range Searching

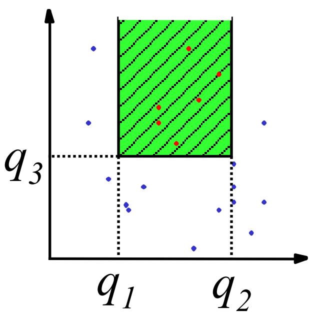
- B-tree solves one-dimensional range searching problem
  - Linear space, $O(\log_B N + T/B)$ query, $O(\log_B N)$ updates

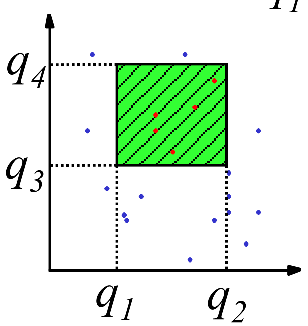


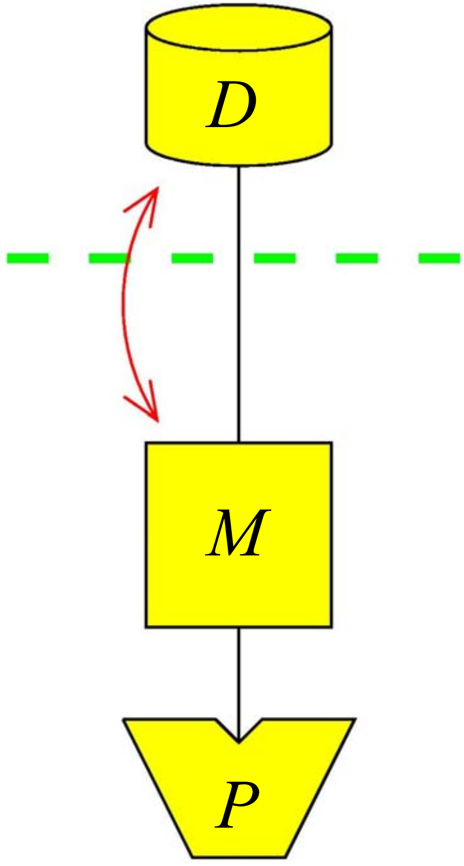
- Cannot be obtained for orthogonal planar range searching:
  - $O(\log_B N + T/B)$ query requires $\Omega(N \frac{\log_B N}{\log_B \log_B N})$ space
  - $O(N)$ space requires $\Omega(\sqrt{N/B} + T/B)$ query

Outline

1. Introduction
2. Fundamental algorithms
3. Buffered data structures
4. Range searching
   – External priority search tree
      * Weight-balanced B-tree
      * Persistent B-trees
   – External Range tree
   – External kd-tree
5. List ranking

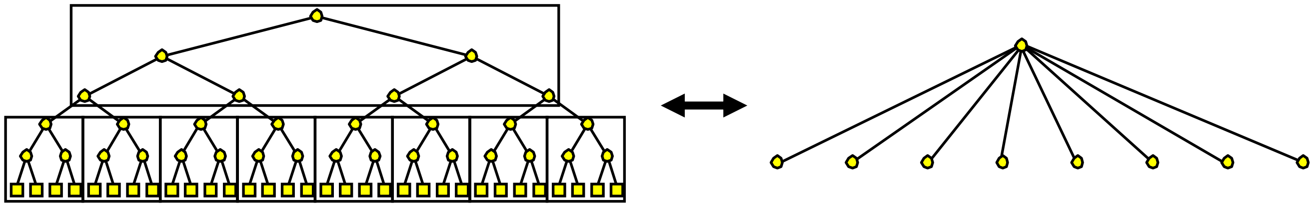






Weight-balanced B-trees

- We will use multilevel structure
  - Attach $O(w(v))$ size structure to weight $w(v)$ node v in B-tree
  - Rebuild secondary structure using $O(w(v))$ I/Os when v split/fuse
- B-tree inefficient since heavy nodes can split/fuse often



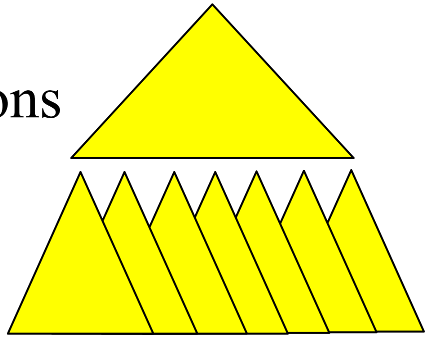
- Weight-balanced B-tree:
  - B-tree but with weight rather than degree balancing constraint
  - Balanced with split/fuse as B-tree

$\Downarrow$

Node v only split/fuse for every $\Omega(w(v))$ updates below it

Persistent B-trees

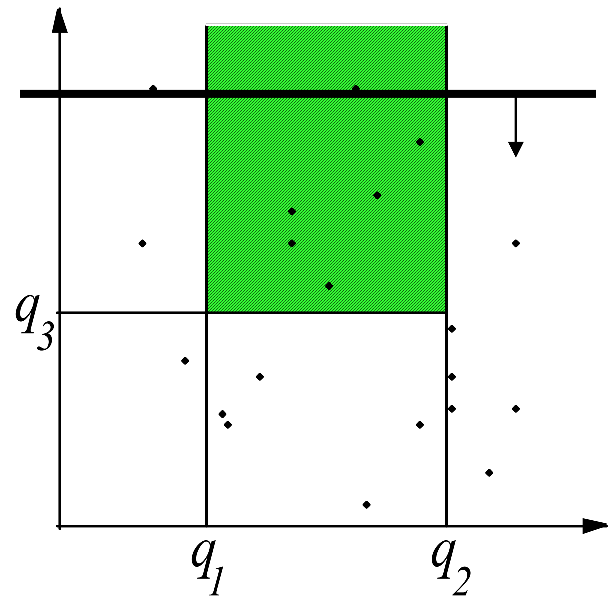
- We will use (partial) persistent B-tree
  - Update current version, query all previous versions



- Partial persistent B-tree (multi-version B-tree) can be obtained using standard techniques
  - $O(\log_B N)$ update, $O(\log_B N + T/B)$ query, $O(N)$ space
  - N is total number of operations performed
  - Batch of N updates in $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/Os using buffer technique
- Idea:
  - Elements and nodes augmented with existence intervals
  - Maintain that every node contains $\Theta(B)$ alive elements in its existence interval

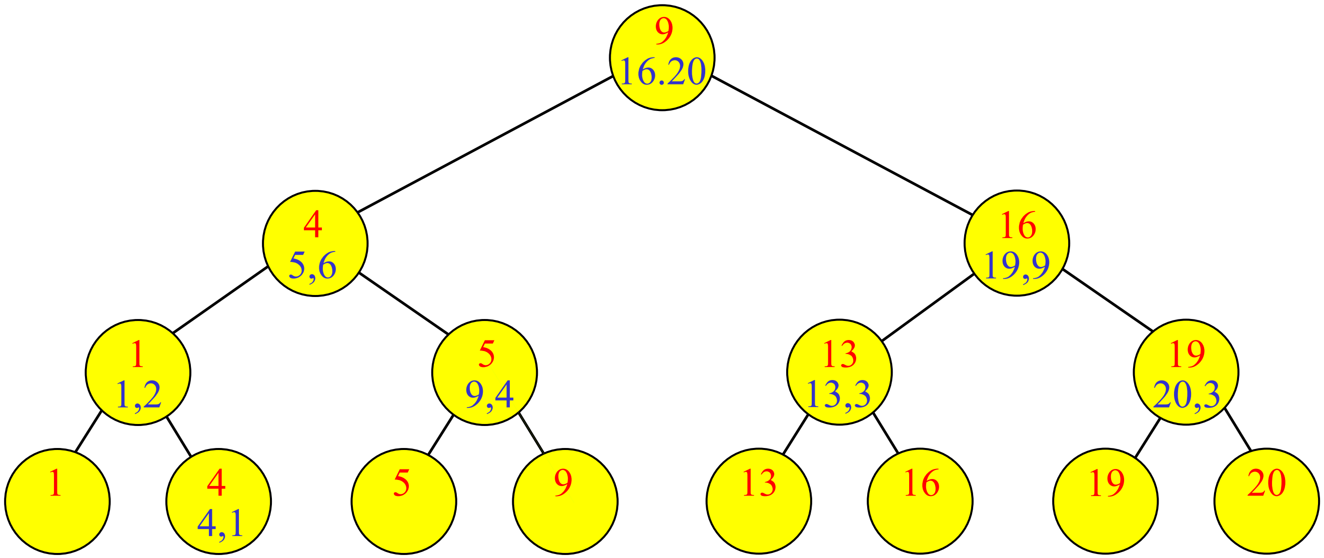
Three-Sided Range Queries

- Report all points (x,y) with $q1 \leq x \leq q2$ and $y \geq q3$
- Static solution:
  - Sweep top-down inserting x in persistent B-tree at (x,y)
  - Answer query by performing range query with $[q_1,q_2]$ in B-tree at q_3



- Optimal:
  - $O(N)$ space
  - $O(\log_B N+T/B)$ query
  - $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ construction
- Dynamic? … in internal memory priority search tree

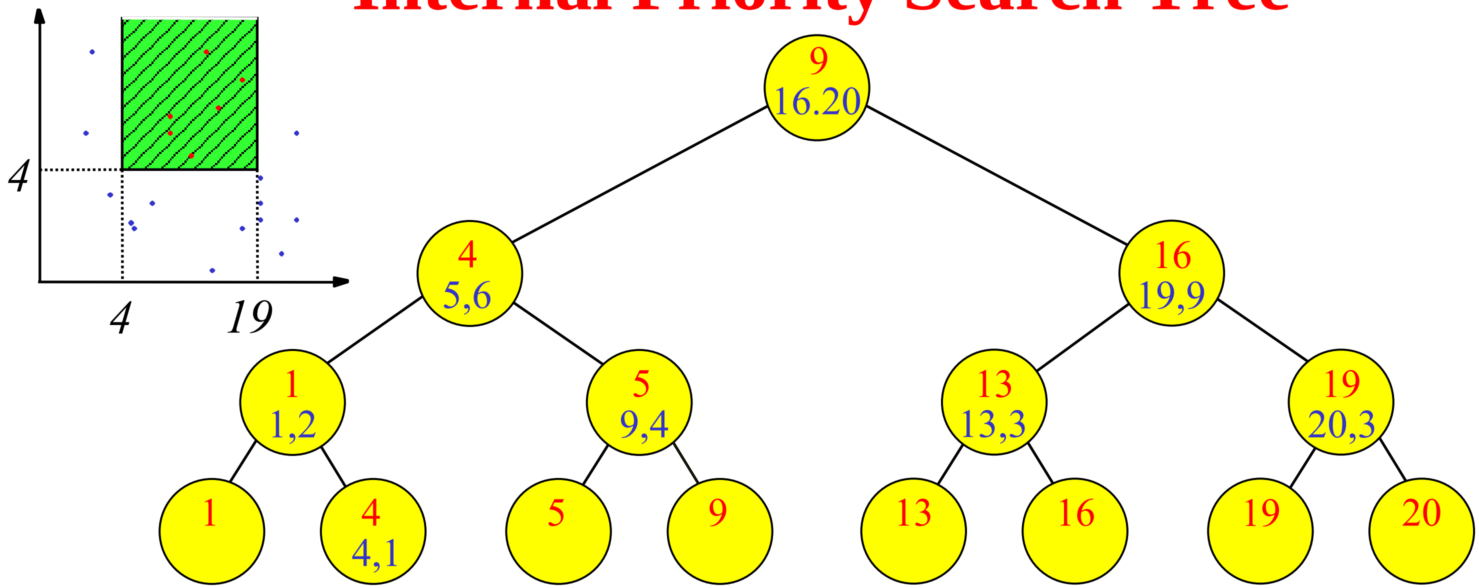
Internal Priority Search Tree



- Base tree on *x*-coordinates with nodes augmented with points
- Heap on *y*-coordinates
  - Decreasing *y* values on root-leaf path
  - (*x*,*y*) on path from root to leaf holding *x*
  - If *v* holds point then *parent*(*v*) holds point

$\Rightarrow$ Linear space and $O(\log N)$ update (traversal of root-leaf path)

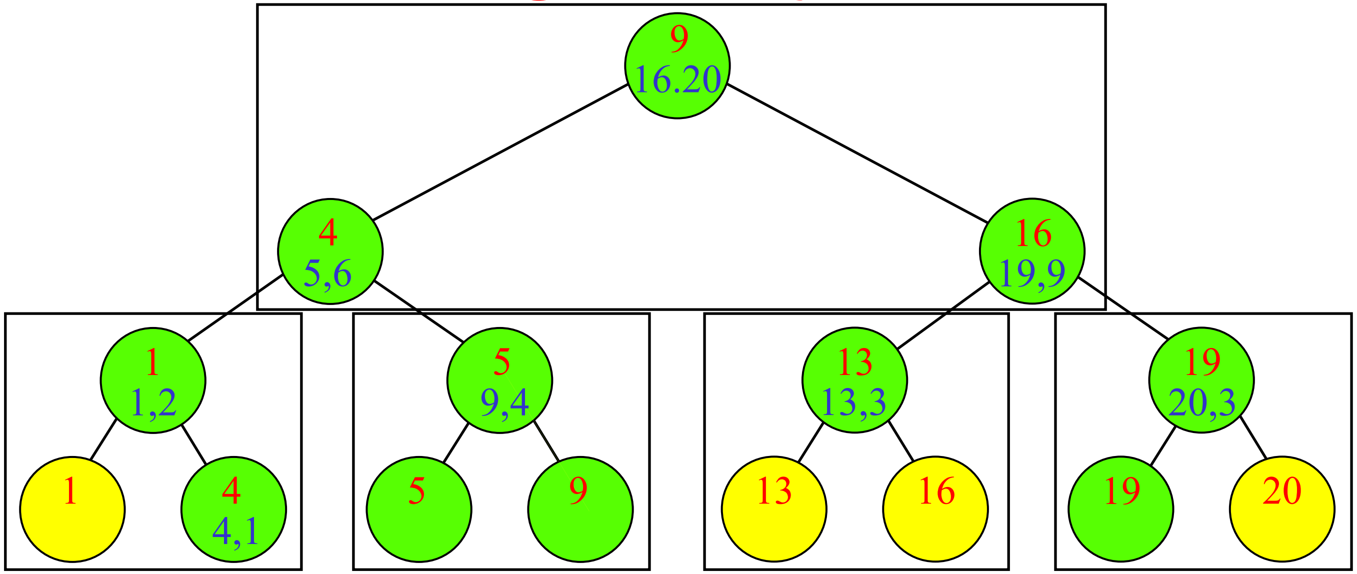
Internal Priority Search Tree



- Query with (q_1, q_2, q_3) starting at root v:
  - Report point in v if satisfying query
  - Visit both children of v if point reported
  - Always visit child(s) of v on path(s) to q_1 and q_2

$\Rightarrow O(\log N+T)$ query

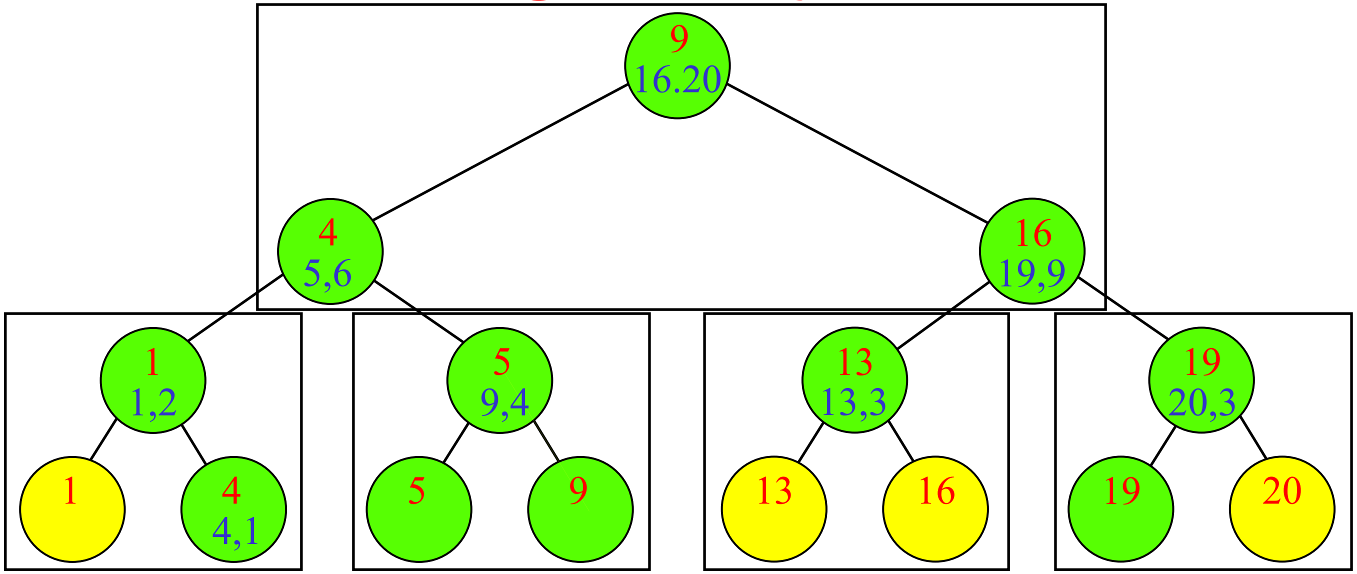
Externalizing Priority Search Tree



- Natural idea: Block tree
- Problem:
  - $O(\log_B N)$ I/Os to follow paths to to q_1 and q_2
  - But $O(T)$ I/Os may be used to visit other nodes ("overshooting")

$\Rightarrow O(\log_B N + T)$ query

Externalizing Priority Search Tree



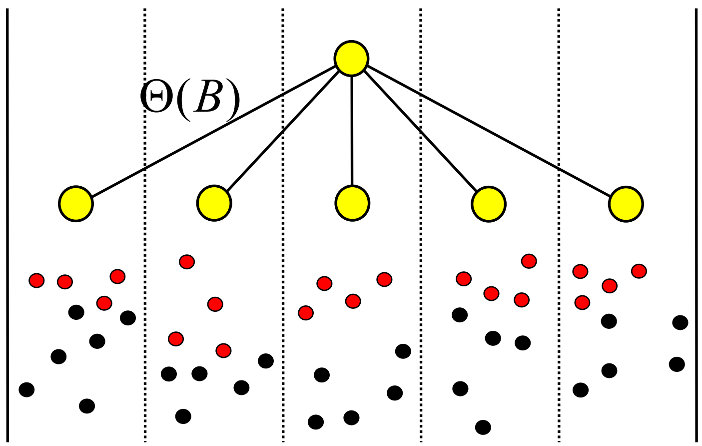
- Solution idea:
  - Store B points in each node $\Rightarrow$
    - * $O(B^2)$ points stored in each supernode
    - * B output points can pay for "overshooting"
  - Bootstrapping:
    - * Store $O(B^2)$ points in each supernode in static structure

External Priority Search Tree

- Base tree: Weight-balanced B-tree on x-coordinates
- Points in "heap order":
  - Root stores B top points for each of the $\Theta(B)$ child slabs
  - Remaining points stored recursively
- Points in each node stored in "$O(B^2)$-structure"
  - Persistent B-tree structure for static problem

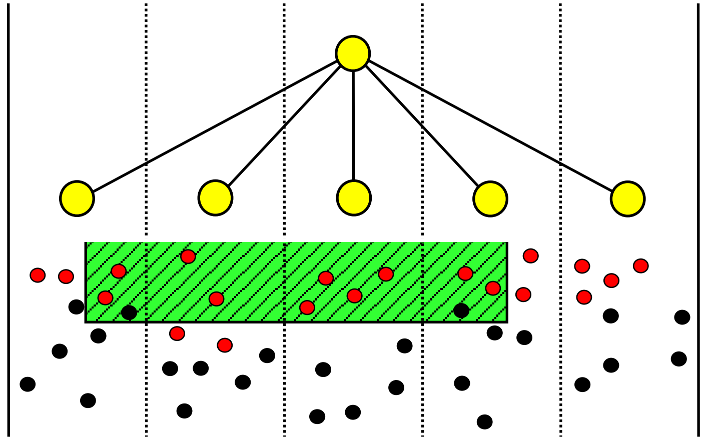
$\Downarrow$

Linear space



External Priority Search Tree

- Query with (q_1, q_2, q_3) starting at root v:
  - Query $O(B^2)$-structure and report points satisfying query
  - Visit child v if
    - * v on path to q_1 or q_2
    - * All points corresponding to v satisfy query

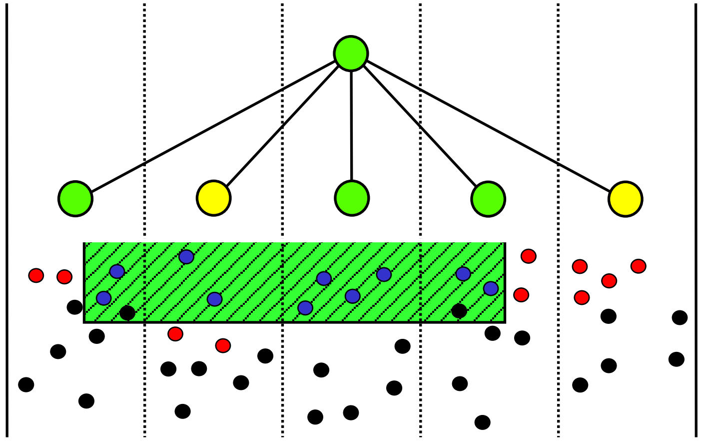


External Priority Search Tree

- Analysis:
  - $O(\log_B B^2 + {}^{T_v}\!/_B) = O(1 + {}^{T_v}\!/_B)$ I/Os used to visit node v
  - $O(\log_B N)$ nodes on path to q_1 or q_2
  - For each node v not on path to q_1 or q_2 visited, B points reported in *parent*(v)

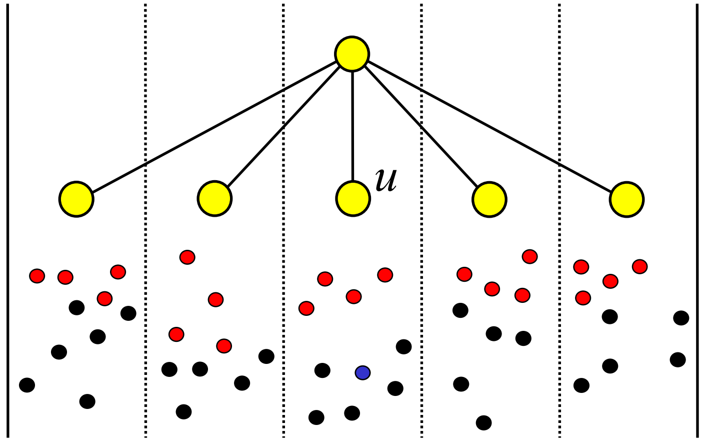
$\Downarrow$

$O(\log_B N + {}^{T}\!/_B)$ query



External Priority Search Tree

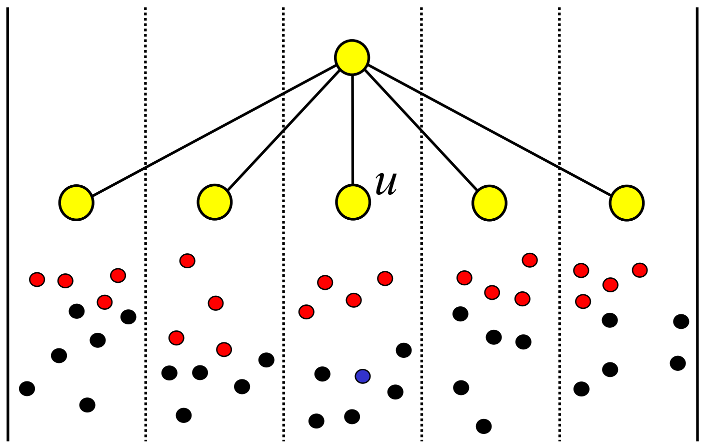
- Insert (x,y) (ignoring insert in base tree - rebalancing):
  - Find relevant node v:
    - * Query $O(B^2)$-structure to find B points in root corresponding to node u on path to x
    - * If y smaller than y-coordinates of all B points then recursively search in u



  - Insert (x,y) in $O(B^2)$-structure of v
  - If $O(B^2)$-structure contains $>B$ points for child u, remove lowest point and insert recursively in u
- Delete: Similarly

External Priority Search Tree

- Analysis:
  - Query visits $O(\log_B N)$ nodes
  - $O(B^2)$-structure queried/updated in each node
    - One query
    - One insert and one delete
- $O(B^2)$-structure analysis:
  - Query: $O(\log_B B^2 + B/B) = O(1)$
  - Update in $O(1)$ I/Os using update block and global rebuilding in $O(\frac{B^2}{B}\log_{M/B}\frac{B^2}{B}) = O(B)$ I/Os



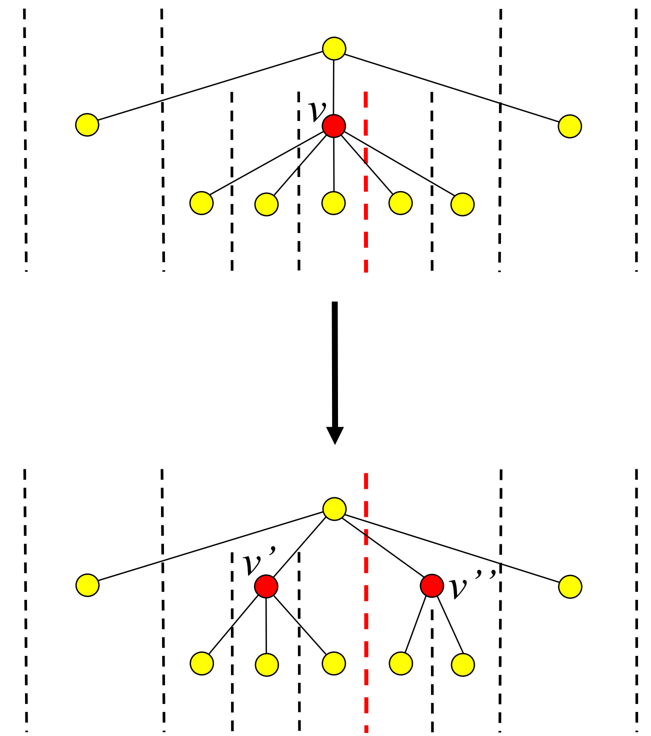
$\Downarrow$

$O(\log_B N)$ I/Os

Dynamic Base Tree

- Deletion:
  - Delete point as previously
  - Delete x-coordinate from base tree using global rebuilding

  $\Rightarrow O(\log_B N)$ I/Os amortized



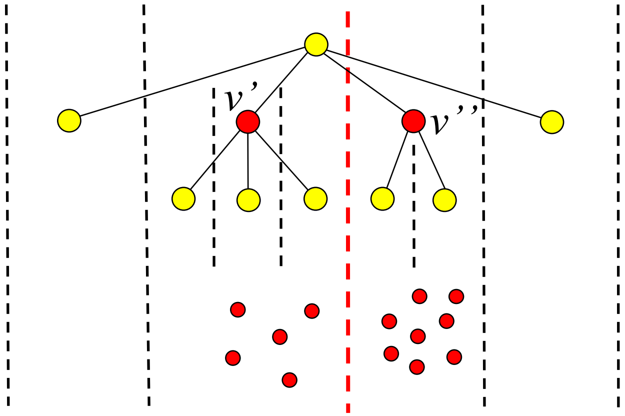
- Insertion:
  - Insert x-coordinate in base tree and rebalance (using splits)
  - Insert point as previously

- Split: Boundary in v becomes boundary in *parent*(v)

Dynamic Base Tree

- Split: When v splits B new points needed in $parent(v)$

- One point obtained from v' (v'') using "bubble-up" operation:
  - Find top point p in v'
  - Insert p in $O(B^2)$-structure
  - Remove p from $O(B^2)$-structure of v'
  - Recursively bubble-up point to v'
- Bubble-up in $O(\log_B w(v))$ I/Os
  - Follow one path from v to leaf
  - Uses $O(1)$ I/O in each node

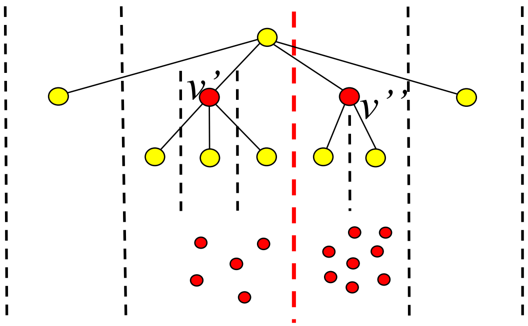


$\Downarrow$

Split in $O(B \log_B w(v)) = O(w(v))$ I/Os

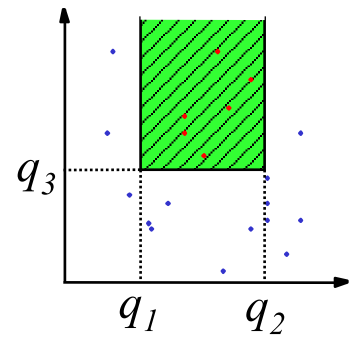
Dynamic Base Tree

- $O(1)$ amortized split cost:
  - Cost: $O(w(v))$
  - Weight balanced base tree: $\Omega(w(v))$ inserts below v between splits



$\Downarrow$

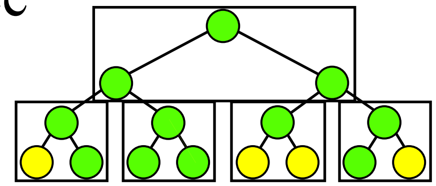
- External Priority Search Tree
  - Space: $O(N)$
  - Query: $O(\log_B N + T/B)$
  - Updates: $O(\log_B N)$ I/Os amortized



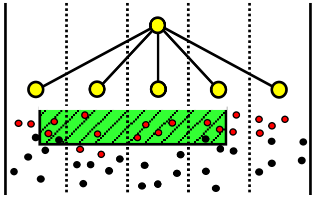
- Amortization can be removed from update bound in several ways
  - Utilizing lazy rebuilding

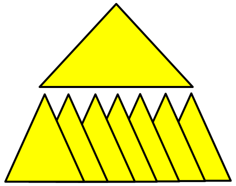
Summary: External Priority Search Tree

- Problem in externalizing internal priority search tree
  - Large fanout and "overshooting"



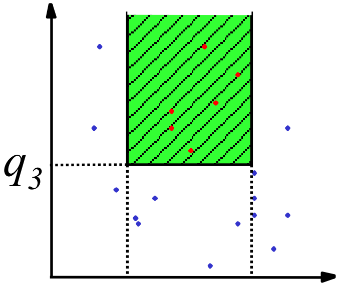
- Solution
  - B^2 points in each node
  - Bootstrapping with persistent B-tree
  - Dynamization using weight-balanced B-tree





$\Downarrow$

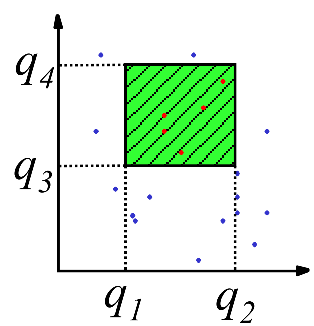
$O(\log_B N + T/B)$ query, $O(\log_B N)$ update

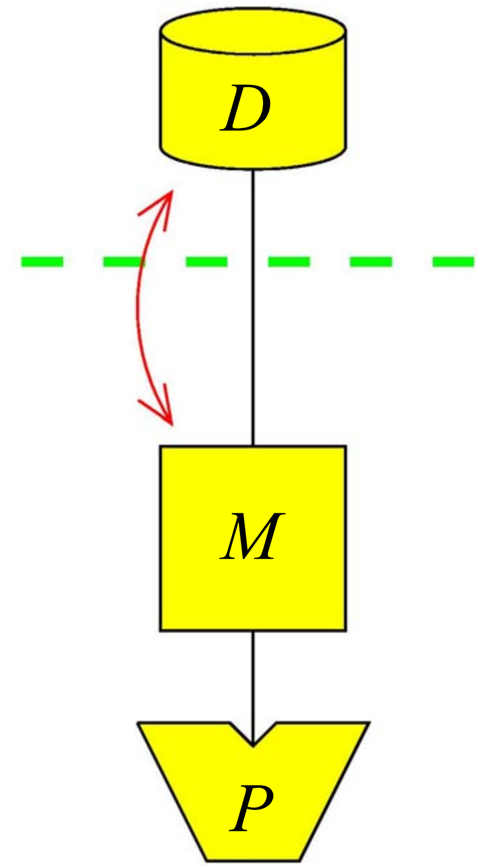


Refs: [A] sec. 3-4, 7

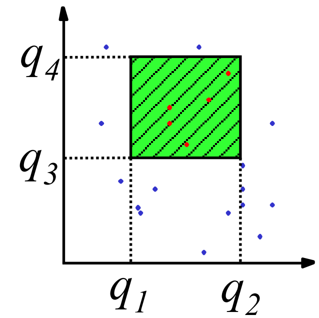
Outline

1. Introduction
2. Fundamental algorithms
3. Buffered data structures
4. Range searching
   – External priority search tree
     * Weight-balanced B-tree
     * Persistent B-trees
   – External Range tree
   – External kd-tree
5. List ranking

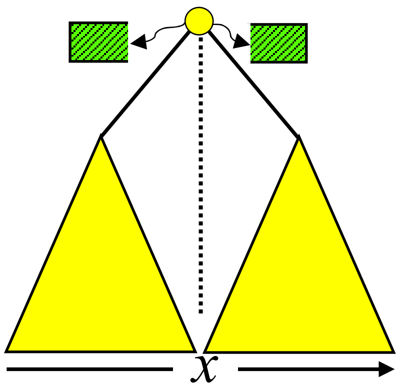




External Range Tree



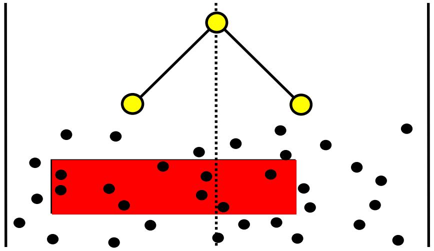
- Structure:
  - Binary base tree on *x*-coordinates (blocked as B-tree)
  - Two priority search trees for 3-sided queries in each node *v* on points below *v*



$\Downarrow$

$O(N \log N)$ space

- Query:
  - Search for top node *v* with q_1 and q_2 below different children
  - Answer 3-sided queries in children of *v*



$\Downarrow$

$O(\log_B N + T/B)$ query

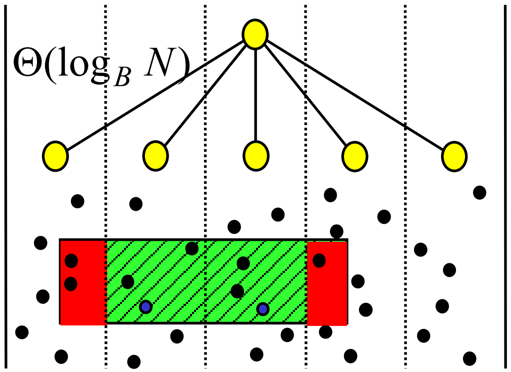
External Range Tree

- Increased fanout to $\Theta(\log_B N)$

  $\Rightarrow$ Space improved to $O(N \log_{\log_B N} N) = O(N \frac{\log_B N}{\log_B \log_B N})$

- Extra external priority search tree in each node
  - to find bottom relevant point in $O(\log_B N)$ slabs spanned by query

  $\Rightarrow$ Query answered in $O(\log_B N + T/B)$ I/Os



- Dynamic with $O(\frac{\log_B^2 N}{\log_B \log_B N})$ update bound using weight-balanced tree

Refs: [A] sec. 8.1

Outline

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   – External priority search tree
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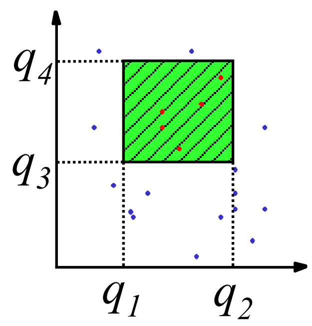
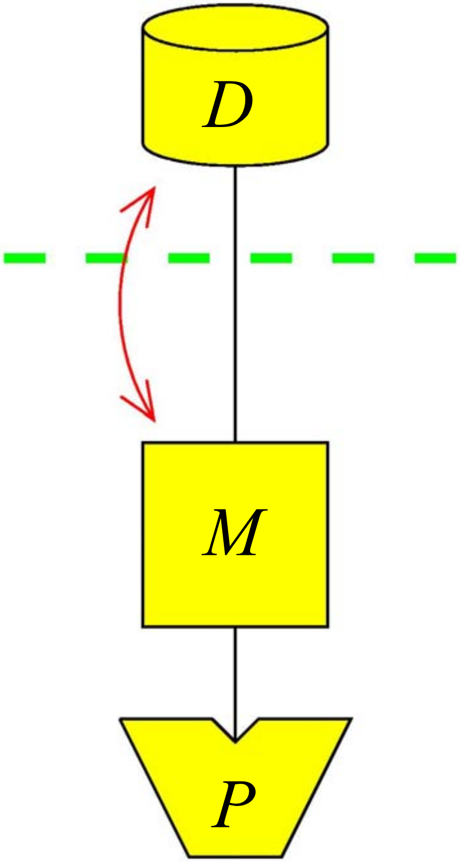
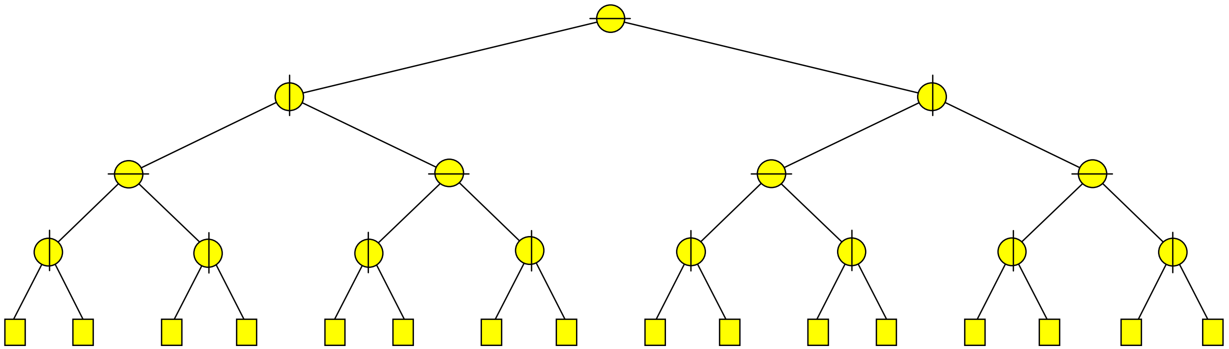
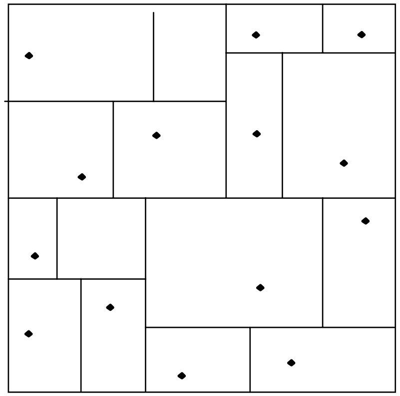
      * Persistent B-trees
   – External Range tree
   – External kd-tree
5. List ranking





External kd-tree





- kd-tree:
  - Recursive subdivision of point-set into two half using vertical/horizontal line
  - Horizontal line on even levels, vertical on uneven levels
  - One point in each leaf

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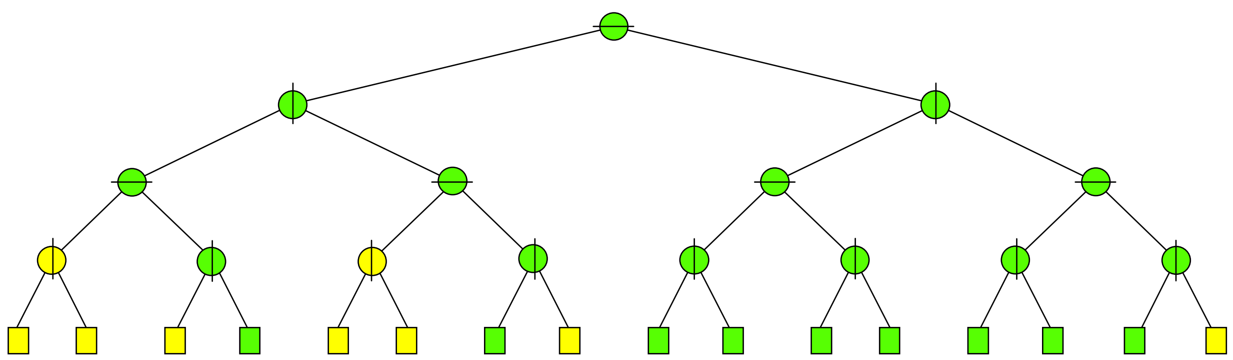
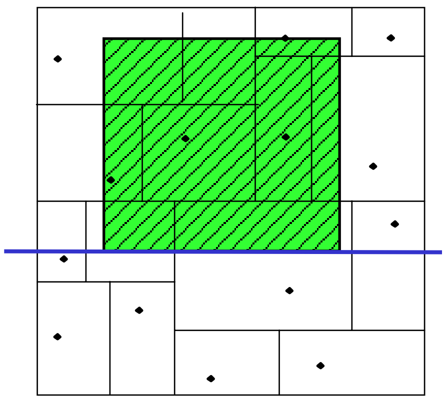
Linear space and logarithmic height

External kd-Tree

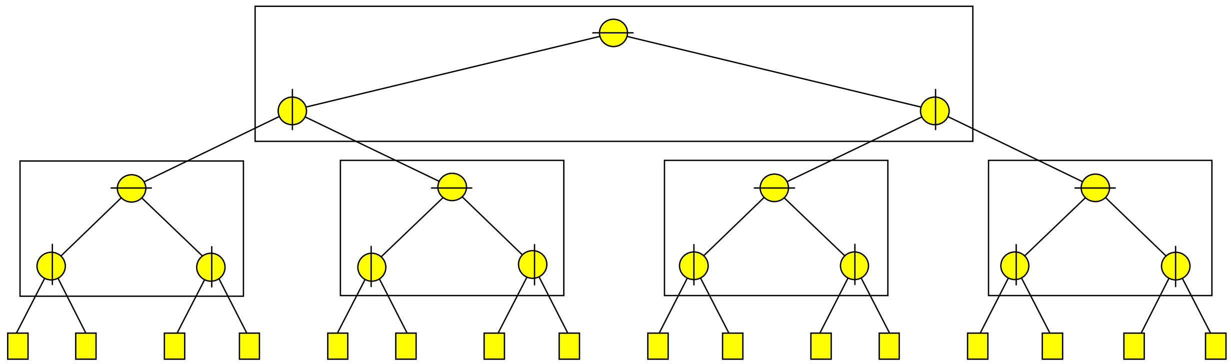




- kd-tree Query
  - Recursively visit nodes corresponding to regions intersecting query
  - Report point in trees/nodes completely contained in query
- kd-tree Query analysis
  - Horizontal line intersect $Q(N) = 2+2Q(N/4) = O(\sqrt{N})$ regions
  - Query covers T regions

  $\Rightarrow O(\sqrt{N} + T)$ I/Os worst-case

External kd-tree

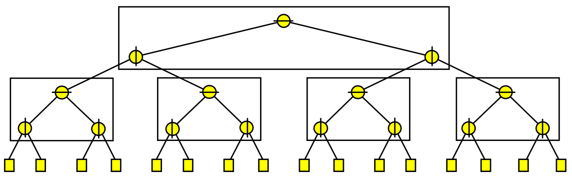


- External kd-tree:
  - Blocking of kd-tree but with B point in each leaf
- Query as before
  - Analysis as before except that each region now contains B points

  $\Rightarrow O(\sqrt{N/B} + T/B)$ I/O query
- Dynamic:
  - Deletes relatively easily in $O(\log_B^2 N)$ I/Os using global rebuilding
  - Insertions also in $O(\log_B^2 N)$ I/Os using logarithmic method

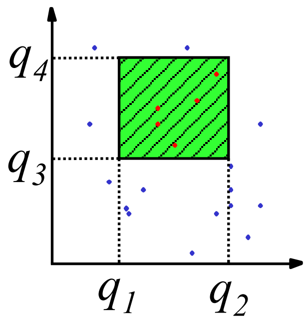
Summary: External kd-tree

- Basically kd-tree with B points in each leaf
  - Updates using logarithmic method



$\Downarrow$

$O(N)$ space, $O(\sqrt{N/B} + T/B)$ query, $O(\log_B^2 N)$ update

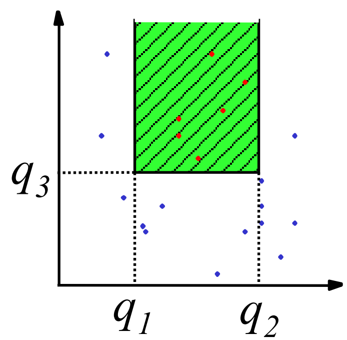


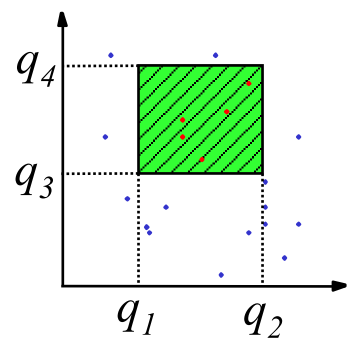
- Update bound can be improved to $O(\log_B N)$ using O-trees
- Easily extended to d-dimensions with $O((N/B)^{1-1/d} + \frac{T}{B})$ query bound

Refs: [A] sec. 8.2

Summary: 3 and 4-sided Range Search

- 3-sided 2d range searching: External priority search tree
  - $O(\log_B N + T/B)$ query, $O(N)$ space, $O(\log_B N)$ update

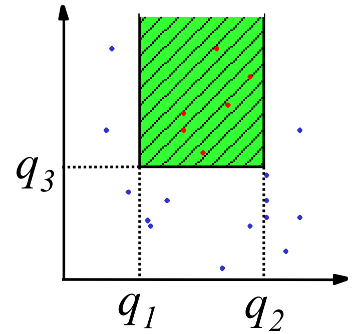




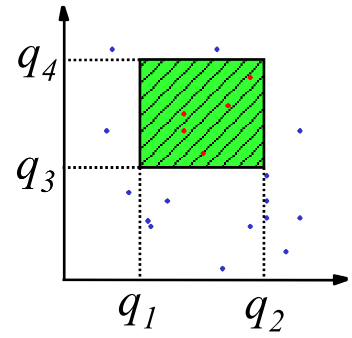
- General (4-sided) *2d* range searching:
  - External range tree: $O(\log_B N + T/B)$ query, $O(N \frac{\log_B N}{\log_B \log_B N})$ space, $O(\frac{\log_B^2 N}{\log_B \log_B N})$ update
  - O-tree: $O(\sqrt{N/B} + T/B)$ query, $O(N)$ space, $O(\log_B N)$ update

Range Searching Tools and Techniques

- Tools:
  - B-trees
  - Persistent B-trees
  - Buffer trees
  - Weight-balanced B-trees
  - Global rebuilding



- Techniques:
  - Bootstrapping
  - Filtering

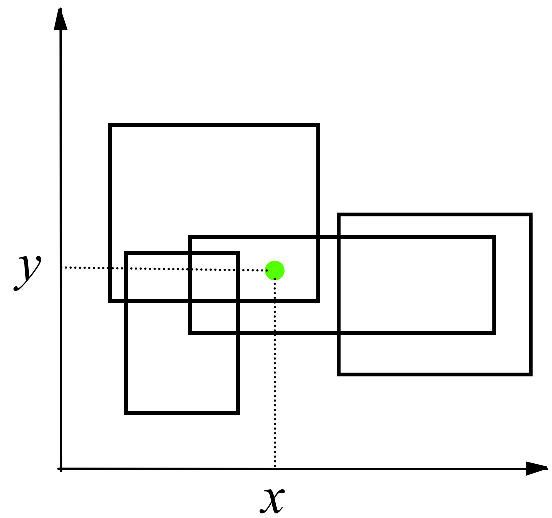


Other Data Structure Results

- Many other results for e.g.
  - Higher dimensional range searching
  - Range counting, range/stabbing max, and stabbing queries
  - Halfspace (and other special cases) of range searching
  - Queries on moving objects
  - Proximity queries (closest pair, nearest neighbor, point location)
  - Structures for objects other than points (bounding rectangles)
- Many heuristic structures in database community
- Implementation efforts:
  - LEDA-SM (MPI)
  - STXXL (Karlsruhe)
  - TPIE (Duke/Aarhus)

Point Enclosure Queries

- Dual of 2d range searching problem
  - Report all rectangles containing query point (x,y)



- Internal memory:
  - Can be solved in O(N) space and $O(\log N + T)$ time

Point Enclosure Queries

- Similarity between internal and external results (*space, query*)

	Internal	External
1d range search	$(N, \log N + T)$	$(N, \log_B N + T/B)$
3-sided 2d range search	$(N, \log N + T)$	$(N, \log_B N + T/B)$
2d range search	$\left(N, \sqrt{N} + T\right)$ $\left(N \frac{\log N}{\log\log N}, \log N + T\right)$	$\left(N, \sqrt{N/B} + T/B\right)$ $\left(N \frac{\log_B N}{\log_B \log_B N}, \log_B N + T/B\right)$
2d point enclosure	$(N, \log N + T)$	$(N, \log N + T/B)$ $(N, \log_B N + T/B)$? $(NB^{\varepsilon}, \log_B N + T/B)$

  - in general tradeoff between space and query I/O

Outline

1. Introduction
2. Fundamental algorithms
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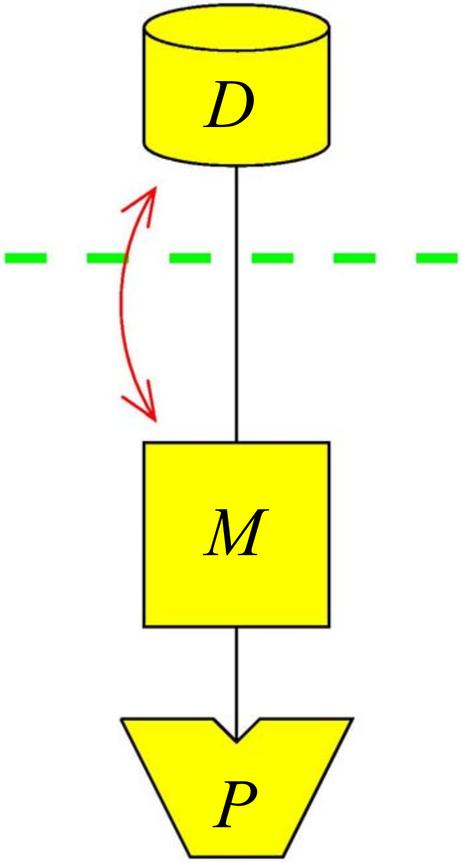
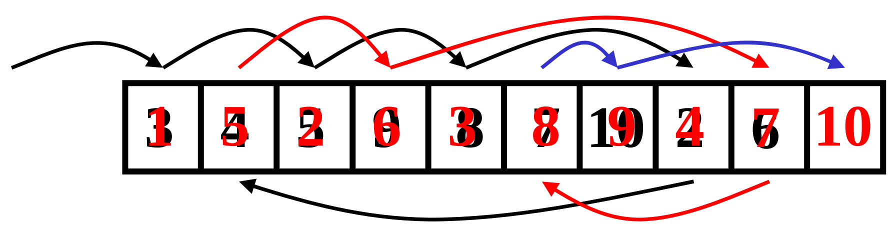
4. Range searching
5. List ranking



List Ranking

- Problem:
  - Given N-vertex linked list stored in array
  - Compute rank (number in list) of each vertex

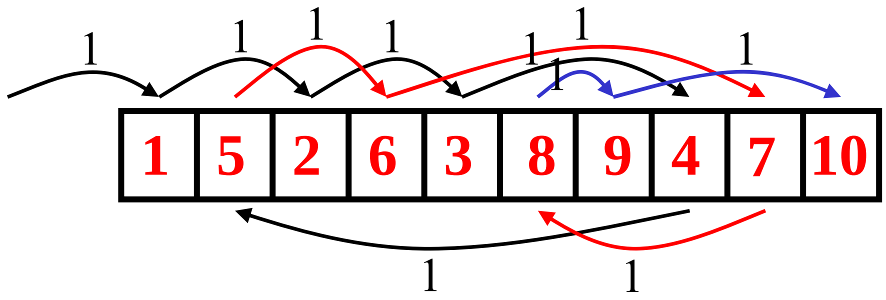


- One of the simplest graph problem one can think of
- Straightforward $O(N)$ internal algorithm
  - Also use $O(N)$ I/Os in external memory
- Much harder to get $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ external algorithm

List Ranking

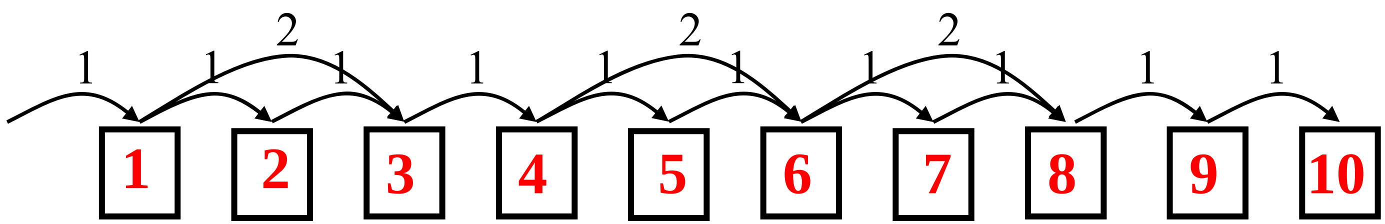
- We will solve more general problem:
  - Given N-vertex linked list with edge-weights stored in array
  - Compute sum of weights (rank) from start for each vertex

- List ranking: All edge weights one



- Note: Weight stored in array entry together with edge (next vertex)

List Ranking

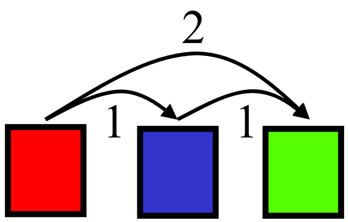


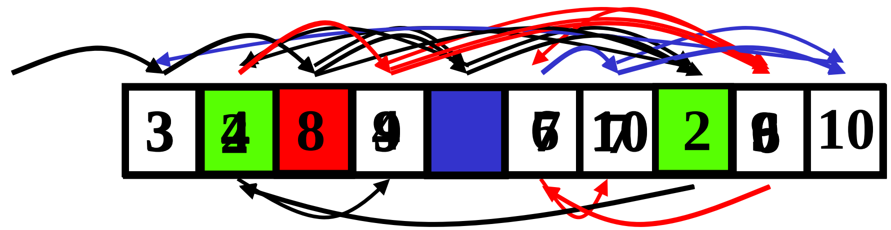
- Algorithm:
  1. Find and mark independent set of vertices
  2. "Bridge-out" independent set: Add new edges
  3. Recursively rank resulting list
  4. "Bridge-in" independent set: Compute rank of independent set

- Step 1, 2 and 4 in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os
- Independent set of size αN for $0 < \alpha \leq 1$

$$\Rightarrow T(N) = T((1-\alpha)N) + O(\frac{N}{B}\log_{M/B}\frac{N}{B}) = O(\frac{N}{B}\log_{M/B}\frac{N}{B}) \text{ I/Os}$$

List Ranking: Bridge-out/in





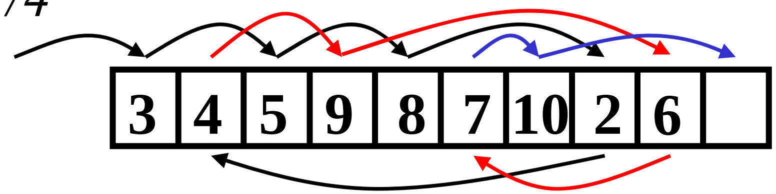
- Obtain information (edge or rang) of successor
  - Make copy of original list
  - Sort original list by successor id
  - Scan original and copy together to obtain successor information
  - Sort modified original list by id

$\Rightarrow O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/Os

List Ranking: Independent Set

- Easy to design $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ randomized algorithm:
  - Scan list and flip a coin for each vertex
  - Independent set is vertices with head and successor with tails

$\Rightarrow$ Independent set of expected size *N/4*



- Deterministic algorithm:
  - 3-color vertices (no vertex same color as predecessor/successor)
  - Independent set is vertices with most popular color

$\Rightarrow$ Independent set of size at least *N/3*

- $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ 3-coloring $\Rightarrow$ $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/O algorithm

List Ranking: 3-coloring

- Algorithm:
  - Consider forward and backward lists (heads/tails in two lists)
  - Color forward lists (except tail) alternately red and blue
  - Color backward lists (except tail) alternately green and blue

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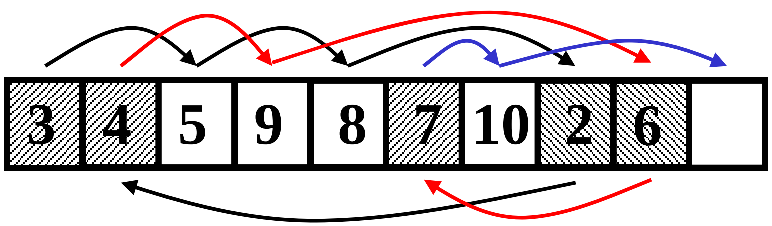
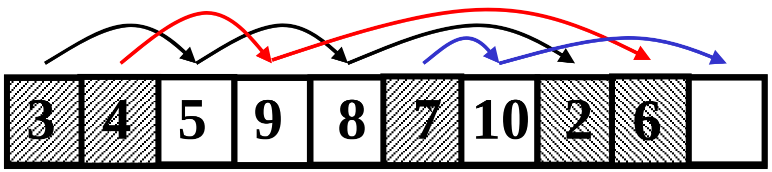
3-coloring



List Ranking: Forward List Coloring

- Identify heads and tails
- For each head, insert red element in priority-queue (priority=position)
- Repeatedly:
  - Extract minimal element from queue
  - Access and color corresponding element in list
  - Insert opposite color element corresponding to successor in queue

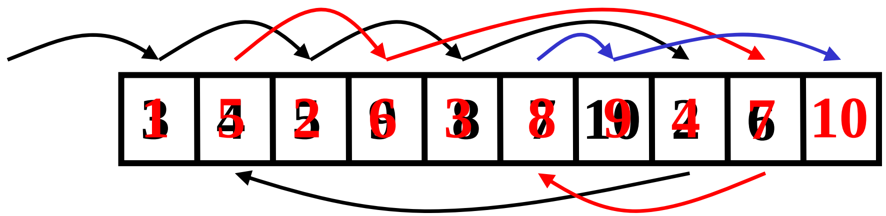


- Scan of list
- O(N) priority-queue operations

$\Rightarrow O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

Summary: List Ranking

- Simplest graph problem: Traverse linked list

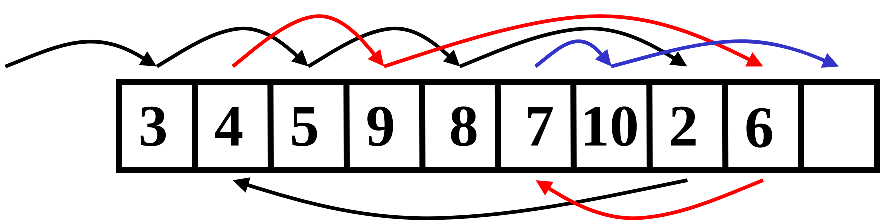


- Very easy $O(N)$ algorithm in internal memory
- Much more difficult $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ external memory
  - Finding independent set via 3-coloring
  - Bridging vertices in/out
- Permuting bound $O(\min\{N, \frac{N}{B}\log_{M/B}\frac{N}{B}\})$ best possible
  - Also true for other graph problems

Refs: [Z] sec. 2, 4.2

Summary: List Ranking

- External list ranking algorithm similar to PRAM algorithm
  - Sometimes external algorithms by "PRAM algorithm simulation"
- Forward list coloring algorithm example of "time forward processing"
  - Use external priority-queue to send information "forward in time" to vertices to be processed later



Other Graph Algorithm Results

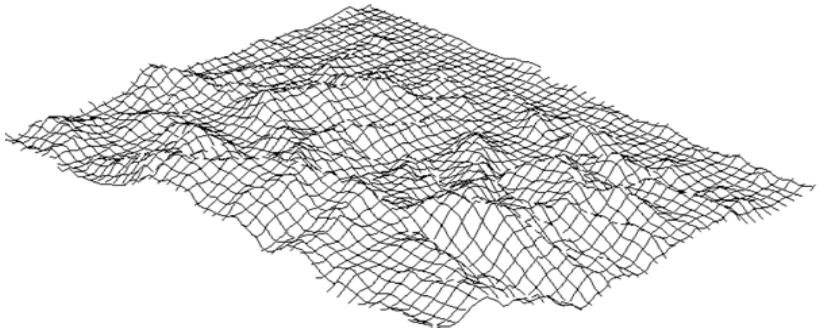
- Most tree problems solved in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os
- Most planar graph problems solved in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

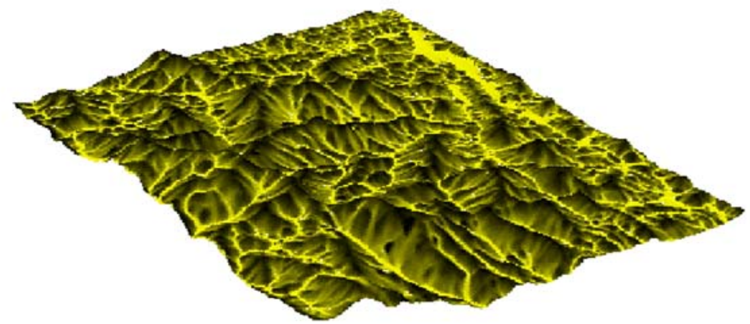
- Most other problems on general graphs not satisfactory solved
  - Directed DFS/BFS: $O((V+E/B)\log_2 V)$ or $O(V+\frac{E}{B}\frac{V}{M})$
  - Undirected BFS: $O(V+\frac{E}{B}\log_{M/B}\frac{E}{B})$ or $O(\sqrt{VE/B}+\frac{E}{B}\log_{M/B}\frac{E}{B})$
  - MSF: $O(V+\frac{E}{B}\log_{M/B}\frac{E}{B})$ or $O(\log_2\log_2\frac{VB}{E}\cdot\frac{E}{B}\log_{M/B}\frac{E}{B})$
  - SSSP: $O(V+\frac{E}{B}\log_2\frac{E}{B})$

- No other than permutation lower bound $O(\min\{E,\frac{E}{B}\log_{M/B}\frac{E}{B}\})$ known

Exercise

Given a grid terrain model (an $\sqrt{N} \times \sqrt{N}$ height grid)





design an $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/O algorithm for computing flow accumulation grid:

- Initially one unit of water in each grid cell
- Water (initial and received) distributed from each cell to lowest lower neighbor cell (if existing)
- Flow accumulation of cell is total flow through it

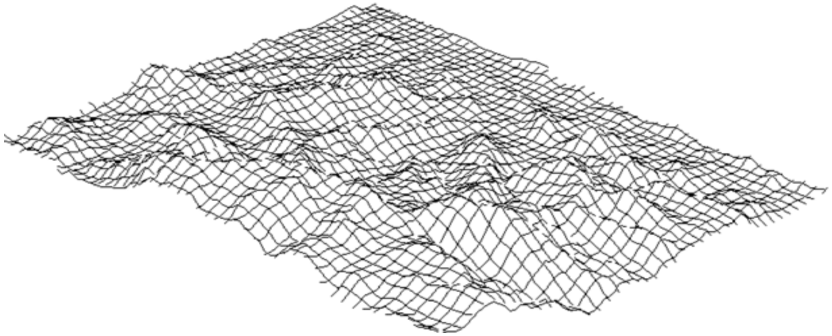
Flow Accumulation

- Problem can easily be solved in $O(N \log N)$ time:

- Process (sweep) points by decreasing height. At each cell:
  - Read flow from flow grid and neighbor heights from height grid
  - Update flow (flow grid) for downslope neighbors

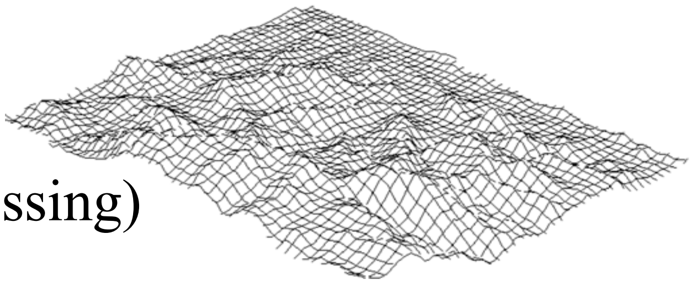
$\Downarrow$

One sweep $\Rightarrow$ $O(N \log N)$ time algorithm

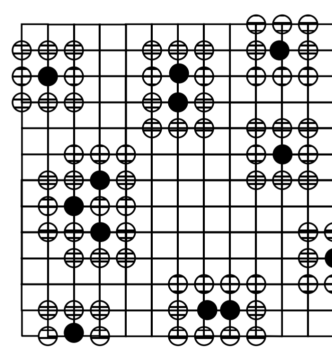


Geometric I/O-bottleneck Example

- Computed for Appalachian Mountains (800km x 800km) by Duke University environmental researchers
  - 100m resolution ⇒ ~ 64M cells
    ⇒ ~128MB raw data (~500MB processing)
    ⇒ 14 days (on 512MB machine)



- Dataset could me much larger:
  - ~ 1.2GB at 30m resolution
    (80% of earth covered by NASA SRTM mission)
  - ~ 12GB at 10m resolution (much of US available)
  - ~ 1.2TB at 1m resolution

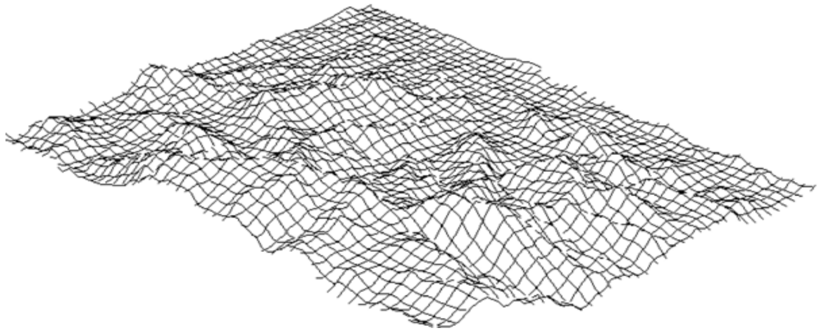


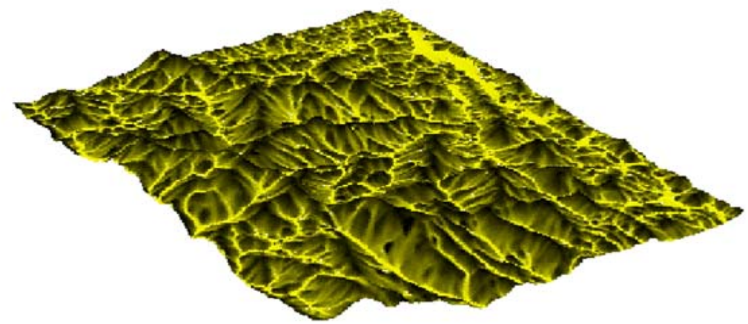
- Problem: Scattered access to grid cells ⇒ $O(N)$ I/Os
- Use of implementation of $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ algorithms

⇒ Appalachian Mountains in 3 hours!

Exercise

Given a grid terrain model (an $\sqrt{N} \times \sqrt{N}$ height grid)



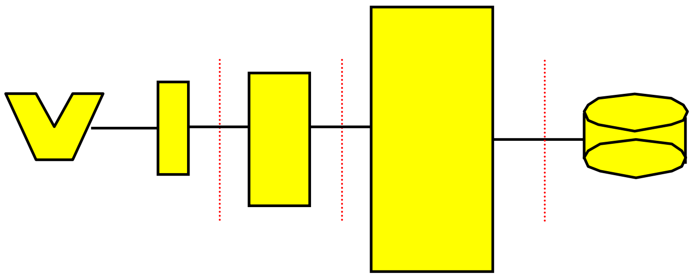


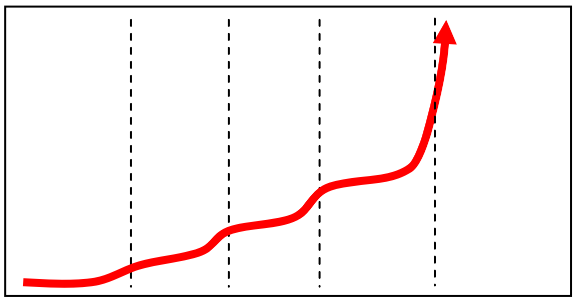
design an $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ I/O algorithm for computing flow accumulation grid:

Hints:

1. Store all neighbor heights with each cell
2. Distribute water to neighbors using time forward processing

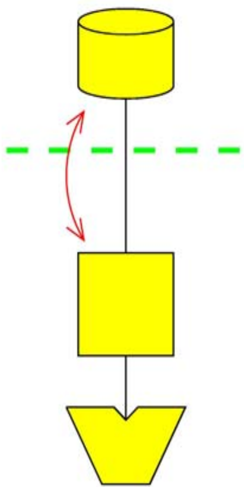
Cache-Oblivious Algorithms





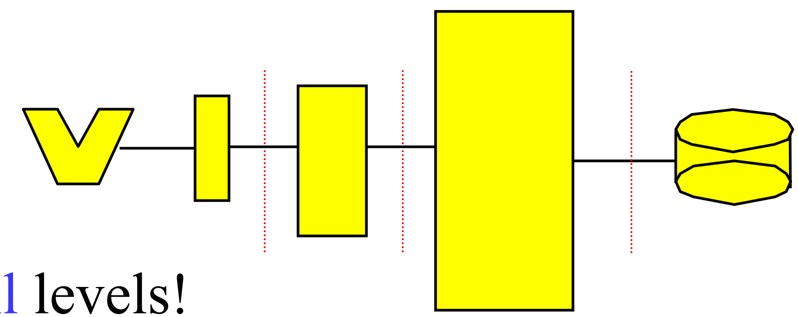
- Block access important on all levels of memory hierarchy
  - But complicated to model whole hierarchy
- I/O-model can be used on all levels
  - But dominating level can change during computation
  - Characteristics of hierarchy may not be known

Cache-Oblivious Algorithms



- *N*, *B*, and *M* as in I/O-model

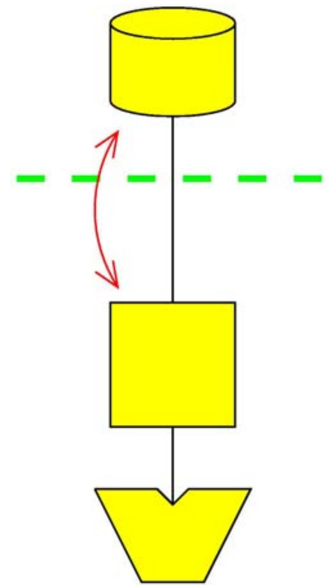
- *M* and *B* not used in algorithm description
- Block transfers (I/O) by optimal paging strategy



Analyze in two-level model
↓
Efficient on one level, efficient of all levels!

- Surprisingly many cache-oblivious algorithms developed recently
  – Much more fundamental work to be done!

Conclusions



- I/O often bottleneck when processing massive data
- Discussed
  - Fundamental algorithms: Sorting and searching
  - Buffered data structures
  - Structures for planar orthogonal range searching
  - List ranking
- Many exciting problems remain open in the area

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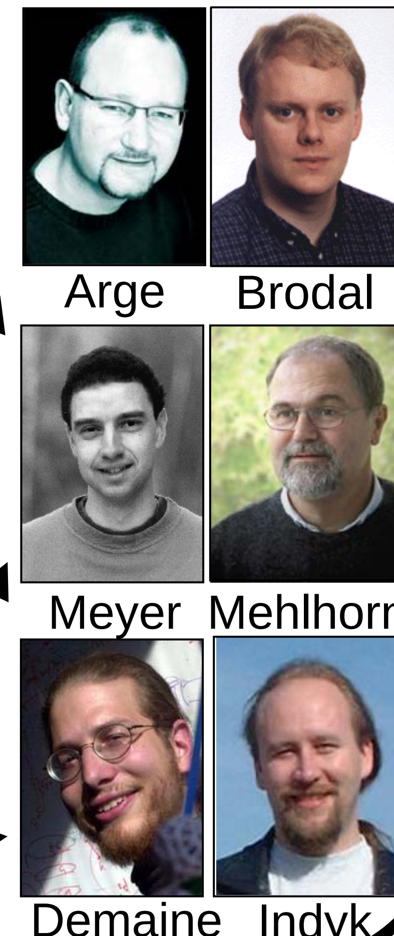
CENTER FOR MASSIVE DATA ALGORITHMICS

- $10M center at University of Aarhus, initially funded for 5 years
- High level objectives:
  - Advance knowledge in massive data algorithms
  - Train researchers in world-leading environment
  - Be catalyst for multidisciplinary collaboration
- Research focus areas:
  - I/O-efficient, streaming, cache-oblivious
  - Algorithm engineering
- Three institution collaboration
  - AU: I/O, cache and algorithm engineering
  - MPI: I/O (graph) and algorithm engineering
  - MIT: Cache and streaming



MADALGO CENTER FOR MASSIVE DATA ALGORITHMICS

Activities

- Exchange of faculty, post docs, students between core institutions
- Short/long visits of faculty, post docs, students from other institutions
- Various workshops
- Symposium on Algorithms for Massive Datasets (yearly from 2008)
- Summer Schools:
  - 2007: Streaming data algorithms
  - 2008: Cache-oblivious algorithms
  - …..

MADALGO CENTER FOR MASSIVE DATA ALGORITHMICS

Summer School

- Data Stream Algorithms: www.madalgo.au.dk/streamschool07
- August 20-23, 2007
- June 15 registration deadline; no registration fee
- Lectures:
  - Sudipto Guha (U. Penn)
  - Sariel Har-Peled (UIUC)
  - Piotr Indyk (MIT)
  - T.S. Jayram (IBM Almaden)
  - Ravi Kumar (Yahoo!)
  - D. Sivakumar (Google)

CENTER FOR MASSIVE DATA ALGORITHMICS

Inauguration

- Inauguration event: www.madalgo.au.dk
- August 24, 2007
- Morning scientific speakers:
  - Jeff Vitter (Purdue): I/O-efficient algorithms
  - Charles Leiserson (MIT): Cache-oblivious algorithms
  - Peter Sanders (Karlsruhe): Algorithm engineering
- Afternoon formal speakers:
  - National Research Foundation chairman Klaus Bock
  - Dean of Science Erik Meineche Schmidt
  - Center Leader Lars Arge

….. and more

- Beer!