What is a J-L transform?

Def:

A distribution $\mathcal{D}$ on real $k \times d$ matrices has the J-L property if

$$\forall x \in \mathbb{R}^d \quad \Pr_{\phi \sim \mathcal{D}} \left[\|\phi x\|_2 \not\approx_\varepsilon \|x\|_2 \right] \le_{c_1} e^{-k\varepsilon^2}$$

$$(0 \le \varepsilon \le 1)$$

Definition not empty (JL '84)

⋮

Nontrivial constructions (AC'06)

TIME($\mathcal{D}$) = # of operations to reduce vector

THM [A. Liberty '07]

For $k < d^{1/2-\delta}$ $\exists \mathcal{D}$ with

$$\text{TIME}(\mathcal{D}) = O(d \lg k)$$

$$k > d^{1/n}$$

THE ALGORITHM

$$r = \frac{1}{\delta}$$

$$\Phi = \underbrace{B D H D^{(r)} \cdots H D^{(2)} H D^{(1)}}$$

$$d \left\{ \underbrace{\begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & 0 \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix}}_{d} \right.$$

$O(d)$-time

$d \times d$ Walsh-Hadamard matrix. $O(d \lg d)$-time

$$H = \frac{1}{\sqrt{d}} \underbrace{\begin{pmatrix} & & \\ & \pm 1\text{'s} & \\ & & \end{pmatrix}}_{d} \Bigg\} d$$

$$B = \frac{1}{\sqrt{k}} \underbrace{\begin{pmatrix} & & \\ & \pm 1\text{'s} & \\ & & \end{pmatrix}}_{d} \Bigg\} k$$

"$B < H$" $\Rightarrow O(d \lg d)$-time

(rows of B are subset of rows of H)

Consider random variable Z:

$$Z = \|BDx\|_2 = \sum_{i=1}^{d} (B^{(i)} x_i) D_i$$

Random ±1

i'th column of B

$$k\left\{\underbrace{\left(\pm 1\text{'s}\right)}_{d}\right.\frac{1}{\sqrt{k}}$$

<u>Def:</u> $M^{(i)} = B^{(i)} x_i$

$$Z = f(D_1, \dots, D_d) = \left\|\sum M^{(i)} D_i\right\|_2$$

$f : \mathbb{R}^d \longrightarrow \mathbb{R}$

f convex

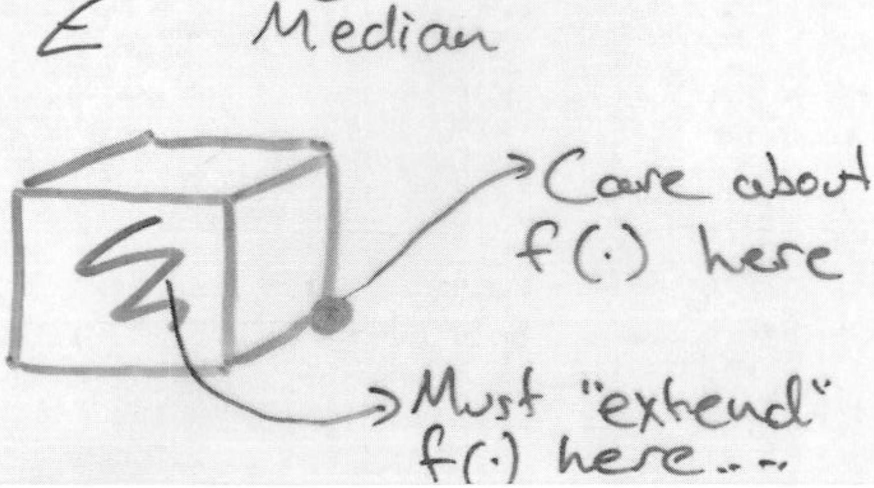
f σ-Lipschitz

$\forall x, y \in \mathbb{R}^d$

$|f(x) - f(y)| \leq \sigma \|x - y\|_2$

(Talagrand)

$$\Longrightarrow \Pr_{D_1 \dots D_d \in \{\pm 1\}}\Big[\,\big|\underbrace{f(D_1, \dots D_d)}_{= Z} - \underbrace{M_f}_{\text{Median}}\big| > \varepsilon\Big] \leq e^{-\frac{\varepsilon^2}{\sigma^2}}$$



Q: What is σ for f ?

A: It is

$$\sigma = \|M\|_{2\to 2}$$

which is

$$\sigma = \|M\|_{2\to 2} = \sup_{\|y\|_2=1} \|y^T M\|_2$$

$$= \sup_{\|y\|_2=1} \sqrt{\sum (y^T M^{(i)})^2}$$

$$= \sup_{\|y\|_2=1} \sqrt{\sum x_i^2 (y^T B^{(i)})^2}$$

Cauchy Schwartz

$$\leq \sup_{\|y\|_2=1} \|x\|_4 \, \|y^T B\|_4$$

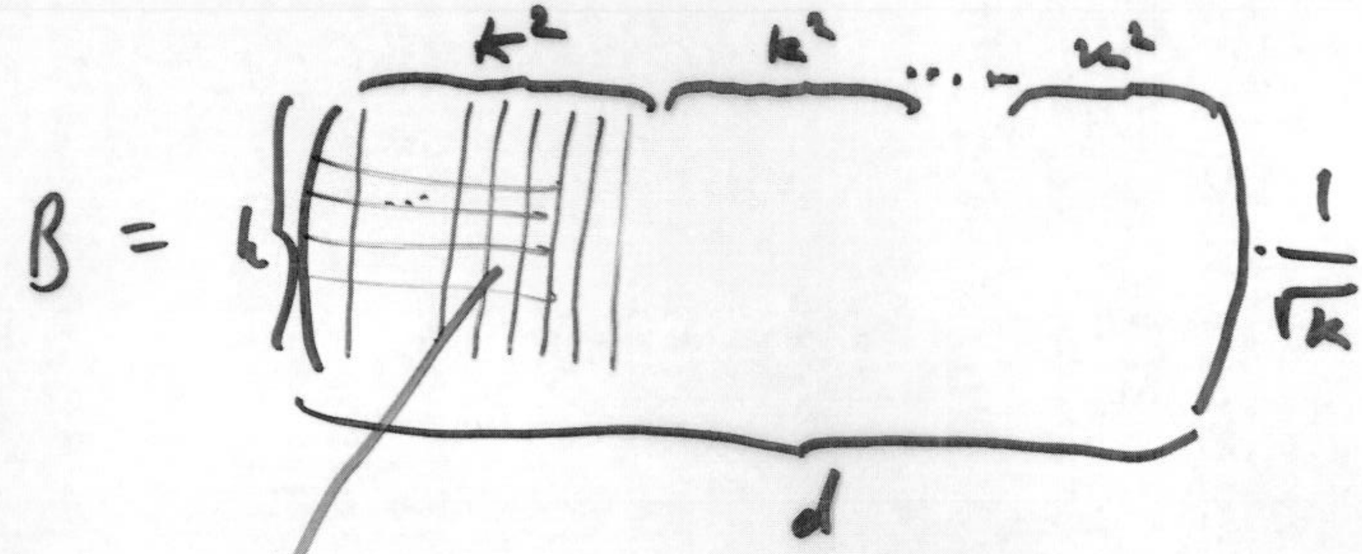
$$= \|x\|_4 \, \|B^T\|_{2\to 4}$$

Q: What is Median of f ?

A: $\approx \|x\|_2 = 1$ (Assumption)

$$\boxed{\Pr\left[\,|f-1| > \varepsilon\,\right] \leq e^{-\frac{\varepsilon^2}{\|x\|_4^2 \, \|B^T\|_{2\to 4}^2}}}$$

Constructing B s.t. $\|B\|_{2\to 4} = \frac{d^{1/4}}{k^{1/2}}$



"dual BCH code of designed distance 5 and length k"

- 4-wise independent $\Rightarrow$ Bound
- "BCH" $\quad$ x random $\Rightarrow \|x\|_4 \approx d^{1/4}$

$$\boxed{\Pr\left[|f-1| > \varepsilon\right] \le e^{\frac{-\varepsilon^2 \cdot k}{\|x\|_4^2 \, d^{1/2}}}}$$

get $\|HD^{(r)} \cdots HD^{(1)} x\|_4 = d^{-1/4}$

$$\Pr\left[\left|\|(BD)(HD^{(r)}) \cdots (HD^{(1)}) x\|_2 - 1\right| > \varepsilon\right] \le e^{-k\varepsilon^2}$$

$(0 \le \varepsilon \le 1)$

Controlling $\|x\|_4$

$$Y = \|HDx\|_4$$

$$\sigma \le \|x\|_4 \|H\|_{\frac{4}{3}\to 4} = \|x\|_4 \, d^{-1/4}$$

$$M_Y = O(d^{-1/4})$$
(median)

$$\underbrace{\|x\|_4}_{=1} \, d^{-\delta/2}$$

Talagrand:

$$\Pr\left[\|HDx\|_4 > d^{-1/4} + t \right] \le e^{\frac{-t^2}{\|x\|_4^2 \, d^{-1/2}}}$$

$$t = \|x\|_4 \, d^{-\delta/2}$$

$$\Pr\left[\|HDx\|_4 > d^{-1/4} + t \right] \le e^{-d^{1/2-\delta}}$$

$$\underset{\text{(Assumption)}}{\le} e^{-k}$$

Repeat... after r times

$$\Pr\left[\|HD^{(r)} \cdots HD^{(1)} x\|_4 > O\left(d^{-1/4} + \|x\|_4 \, d^{-\frac{r\delta}{2}}\right) \right]$$

$$\le r e^{-k}$$

$\Rightarrow$ set $r = 1/\delta$ ...

Lower Bounds

Any nontrivial lower bound for, say, $k = d^{1/2}$ would imply lower bound for FFT which is a major open problem.