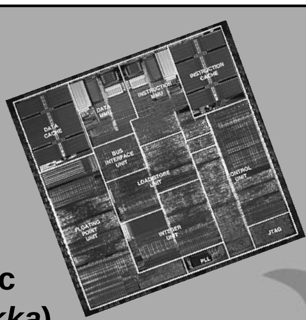
Lecture 6

HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI



Computer Arithmetic (*Tietokonearitmetiikka*)

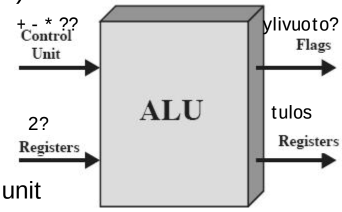
Stallings: Ch 9

Integer representation (*Kokonaislukuesitys*)
Integer arithmetics (*Kokonaislukuaritmetiikka*)
Floating-point representation (*Liukulukuesitys*)
Floating-point arithmetics (*Liukulukuaritmetiikka*)

ALU

- ALU = Arithmetic Logic Unit (*Aritmeettis-looginen yksikkö*)
- Actually performs operations on data
  - Integer and floating-point arithmetic
  - Comparisons (*vertailut*), left and right shifts (*sivuttaissiirrot*)
  - Copy bits from one register to another
  - Address calculations (*Osoitelaskenta*): branch and jump (*hypyt*), memory references (*muistiviittaukset)*
- Data from/to internal registers (latches)
  - Input copied from normal registers (or from memory)
  - Output goes to reg (or memory)
- Operation
  - Based on instruction register, control unit

(Sta06 Fig 9.1)



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Integer representation

(*kokonaislukujen esitys*)

Integer Representation (*Kokonaislukuesitys*)

- Binary representation, bit sequence, only 0 and 1
- ”Weight” of the number based on position

$57 = 5*10^1 + 7*10^0$
$= 32 + 16 + 8 + 1$
$= 1*2^5 + 1*2^4 + 1*2^3 + 0*2^2 + 0*2^1 + 1*2^0$
$= 0011\ 1001$
$=$ 0x39 **hexadecimal**
$= 3*16^1 + 9*16^0$

- Most significant bit, MSB (*eniten merkitsevä bitti*)
- Least significant bit, LSB (*vähiten merkitsevä bitti*)

Integer Representation (Kokonaislukuesitys)

- Negative numbers?
  - Sign magnitude (*Etumerkki-suuruus*)
  - Twos complement (*2:n komplementtimuoto*)

-57 = 1011 1001

-57 = 1100 0111

Sign (*etumerkki*)

- Computers use twos complement
  - Just one zero (no +0 and -0)
    - Comparison to zero easy
  - Math is easy to implement
    - No need to consider sign
    - Subtraction becomes addition
  - Simple hardware and circuit

+2 = 0000 0010
+1 = 0000 0001
0 = 0000 0000
-1 = 1111 1111
-2 = 1111 1110

Twos complement (*2:n komplementti*)

- Example
  - 8-bit sequence, value -57

57 = 0011 1001	unsigned value (*itseisarvo*)
1100 0110	invert bit (ones complement)
1100 0110	
1	add 1
1100 0111	twos complement

Reject overflow

  - Easy to expand. As a 16-bit sequence

57 = 0011 1001 = 0000 0000 0011 1001
-57 = 1100 0111 = 1111 1111 1100 0111

sign extension

Twos complement

- Value range (*arvoalue*): $-2^{n-1} \ldots 2^{n-1} -1$

8 bits: $-2^7 \ldots 2^7-1 = -128 \ldots 127$
32 bits: $-2^{31} \ldots 2^{31}-1 = -2\ 147\ 483\ 648 \ldots 2\ 147\ 483\ 647$

- Addition overflow (*yhteenlaskun ylivuoto*) easy to detect
  - No overflow, if different signs in operands
  - Overflow, if same sign (*etumerkki*) and the results sign differs from the operands

```
   57 = 0011 1001
 + 80 = 0101 0000
 ----------------
  137 = 1000 1001    Overflow!
```


Twos complement

- Subtraction as addition (*vähennyslasku yhteenlaskuna*)!
  - Forget the sign, handle as if unsigned!
  - Complement 2nd term, subtrahend (*2:n komplementti vähentäjästä*) then add
  - Simple hardware

```
+1 =   0001
-3 =   1101
-----------
-2 =   1110
```

3= 0011

```
1100
   1
----
1101
```

-3 in two complement

  - Check
    - Overflow? (same rule as in addition)
    - sign= 1, result is negative

(Sta06 Table 9.1)

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Integer arithmetics (*kokonaislukuaritmetiikkaa*)

Negation (*negaatio*)
Addition (*yhteenlasku*)
Subtraction (*vähennyslasku*)
Multiplication (*kertolasku*)
Division (*jakolasku*)

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Negation = Twos complement

- 1: invert all bits
- 2: add 1
- 3: Special cases
  - Ignore carry bit (*ylivuotobitti*)
  - Sign really changed?
    - Cannot negate smallest negative
    - Result in exception
- Simple hardware

```
-57 = 1100 0111
      0011 1000
              1
      ---------
      0011 1001
= 57
```

```
-128 = 1000 0000
       0111 1111
               1
       ---------
       1000 0000
```

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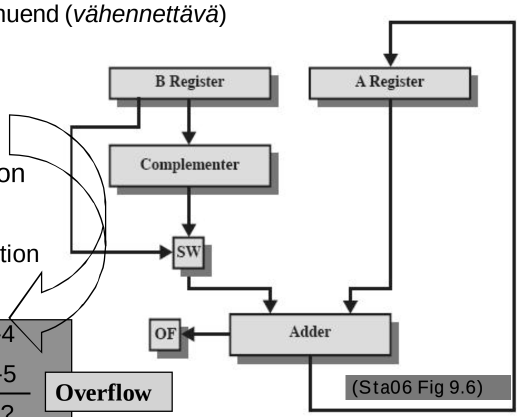
Addition (and subtraction)

- Normal binary addition
  - In subtraction: complement the 2. operand, subtrahend (*vähentäjä*) and add to 1. operand, minuend (*vähennettävä*)
- Ignore carry
  - Check sign!
    Overflow indication
- Simple hardware function
  - Two circuits:
    Complement and addition

1100 =-4	1100 = -4
+1111 =-1	+1011 = -5
11011 =-5	10111 = ?

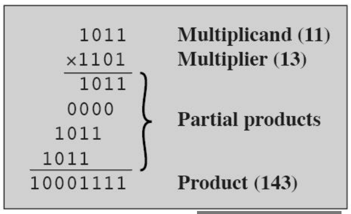
Overflow



(Sta06 Fig 9.6)

Integer multiplication

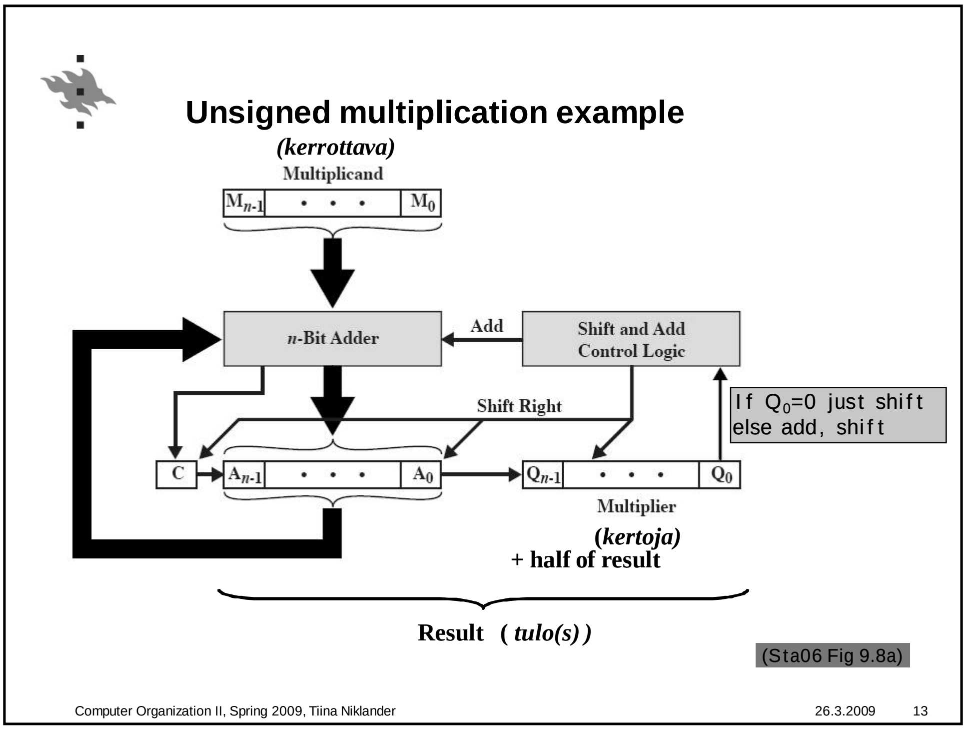
- "Just like" you learned at school
  - Easy with just 0 and 1!
- Hardware?
  - Complex
  - Several algorithms
- Overflow?
  - 32 b operands → result 64 b?
- Simpler, if only unsigned numbers
  - Just multiple additions
  - Or additions and shifts
    - Shift left = multiply by 2
    - esim: 5 * => add, shift, shift, add



(Sta06 Fig 9.7)

```
2* 10011  => 100110
```

```
Example: 5*11
add=> 1011
shift=> 10110
shift=> 101100
add=>110111 (= 55)
```


Unsigned multiplication example
(kerrottava)
Multiplicand
M_{n-1} • • • M_0
n-Bit Adder
Add
Shift and Add Control Logic
Shift Right
If Q_0=0 just shift
else add, shift
C
A_{n-1} • • • A_0
Q_{n-1} • • • Q_0
Multiplier
(kertoja)
+ half of result
Result (tulo(s))
(Sta06 Fig 9.8a)
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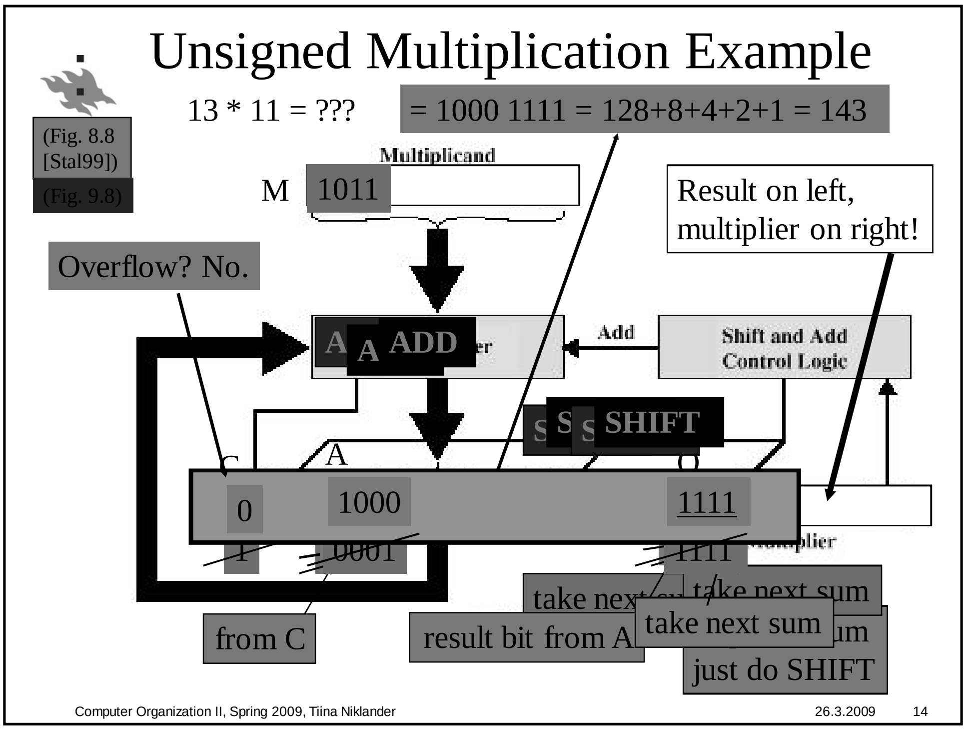
Unsigned Multiplication Example
13 * 11 = ???
= 1000 1111 = 128+8+4+2+1 = 143
(Fig. 8.8 [Stal99])
(Fig. 9.8)
Multiplicand
M 1011
Result on left,
multiplier on right!
Overflow? No.
ADD
Add
Shift and Add Control Logic
SHIFT
A
0
1000
1111
1
0001
1111
from C
result bit from A
take next sum
just do SHIFT
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Unsigned multiplication

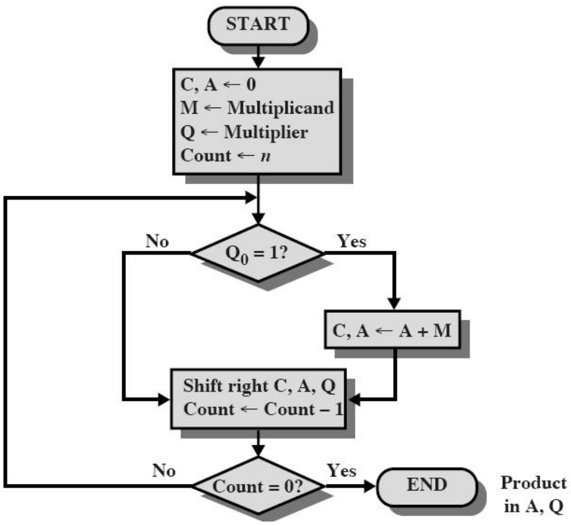
Q * M = 1101 * 1011 = 1000 1111 eli 13*11 = 143

C	A	Q	M		
0	0000	1101	1011	Initial Values	
0	1011	1101	1011	Add	First Cycle
0	0101	1110	1011	Shift	
0	0010	1111	1011	Shift	Second Cycle
0	1101	1111	1011	Add	Third Cycle
0	0110	1111	1011	Shift	
1	0001	1111	1011	Add	Fourth Cycle
0	1000	1111	1011	Shift	

(b) Example from Figure 9.7 (product in A, Q)

(Sta06 Fig 9.8b)

Unsigned multiplication



(Sta06 Fig 9.9)

Multiplication with negative values?

- The preceeding algorithm for unsigned numbers does NOT work for negative numbers
- Could do with unsigned numbers
  - ❶ Change operands to positive values
  - ❷ Do multiplication with positive values
  - ❸ Check signs and negate the result if needed
- This works, but there are better and faster mechanisms available

Booth's Algorithm

- Unsigned multiplication:
  - Addition (only) for every "1" bit in multiplier (*kertoja*)
- Booth's algorithm (improvement)
  - Combine all adjacent 1's in multiplier together,
  - Replace all additions by one subtraction and one addition
  - Example: 7*x = 8*x +(–x)
  - 111*x = 1000*x +(-x) =
  - add, shift, shift, shift, complement, add (in reality, the complement would be first)

5 * 7 = 0101 * 0111
= 0101 * (1000-0001)

➡

00101000	40
11111011	-5
100100011 =	35

- Works for twos complement! Also negative values!

Booth's algorithm

Arithmetic Shift Right:
= fill with sign

1000 1000
1100 0100

Sign bit extending

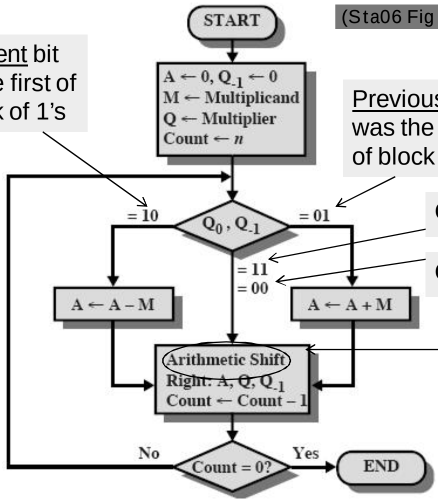
(Sta06 Fig 9.12)

Current bit is the first of block of 1's

Previous bit was the last of block of 1's

Continuing block of 1's

Continuing block of 0's



Why does it work?
M*(011111111) = $2^7 - 1$
M*(00011110) = $2^5 - 2^1$
M*(01111010) = $2^7 - 2^3 + 2^2 - 2^1$

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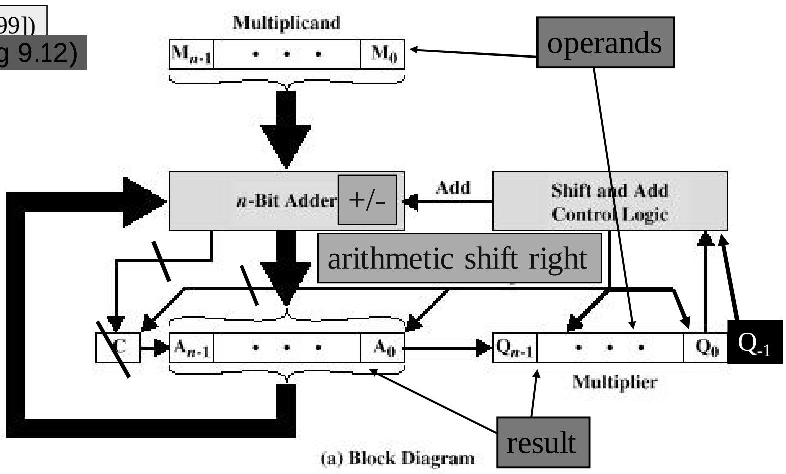
Booth's Algorithm for Twos Complement Multiplication

(Fig. 8.12 [Stal99])
(Sta06 Fig 9.12)



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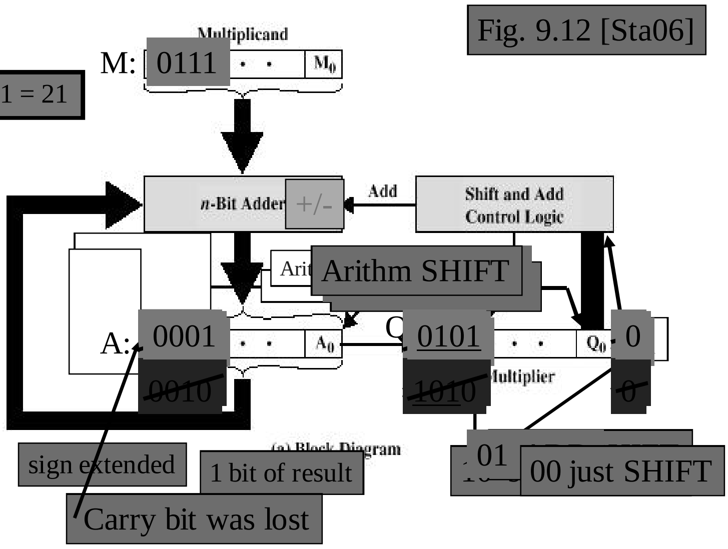
Booth's Algorithm Example

7 * 3 = ?

= 0001 0101 = 21

Fig. 9.12 [Sta06]



Booth's Algorithm, example

Sta06 Fig 9.12

1-0 subtract (*vähennys*)

0-1 add (*lisäys*)

Q * M = 0011 * 0111 = 0001 0101 eli 3*7 = 21

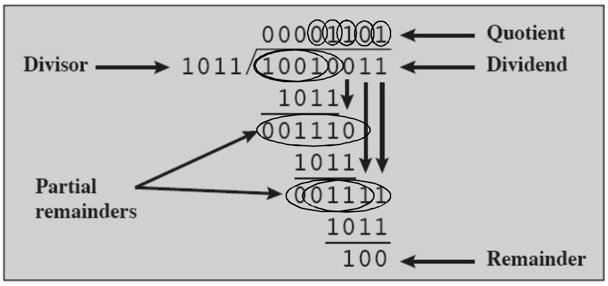
A	Q	Q_{-1}	M		
0000	0011	0	0111	Initial Values	
1001	0011	0	0111	A ← A - M	First Cycle
1100	1001	1	0111	Shift	
1110	0100	1	0111	Shift	Second Cycle
0101	0100	1	0111	A ← A + M	Third Cycle
0010	1010	0	0111	Shift	
0001	0101	0	0111	Shift	Fourth Cycle

(Sta06 Fig 9.13)

Integer division (*kokonaislukujen jakolasku*)

- Like in school algorithm
  - Easy: new quotient digit always 0 or 1

(jakaja)

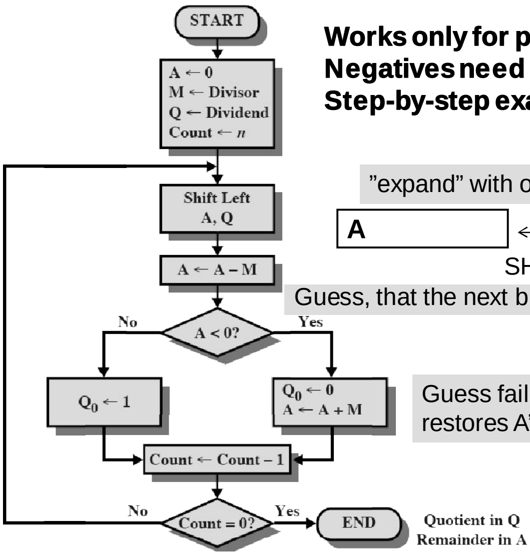


(osamäärä)
(jaettava)
(jakojäännös)

- Hardware needs as in multiplication
  - Shift left = consider new digit

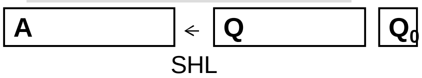
(Sta06 Fig 9.15)

Integer division (*kokonaislukujen jakolasku*)



Works only for positive integers,
Negatives need some extra work,
Step-by-step example Fig 9.17 [Sta06]

"expand" with one more digit



Guess, that the next bit is 1

Guess failed,
restores A's previous value

Sta06 Fig 9.16

Example: twos complement division

- Division: 7/3 A+ Q = 7 = 0000 0111 M= 3 = 0011

A	Q	
0000	0111	initial value
0000	1110	shift left
1101		subtract M
0000	1110	restore
0001	1100	shift left
1110		subtract M
0001	1100	restore
0011	1000	shift left
0000		subtract M
0000	1001	set Q_0=1
0001	0010	shift
1110		subtract M
0001	0010	restore

Sta06 Fig 9.17 a

Subtract M = Add (–M)
-M = -3 = 1101

First try, if you can do the subtraction (or add if different signs).
If the sign changed, subtraction failed and A must be restored, Q_0 = 0

If subtraction succesful, Q_0 = 1

Q = quotient = 2
A = remainder = 1

Repeat as many times as Q has bits.

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Floating Point Representation (*Liukulukuesitys*)

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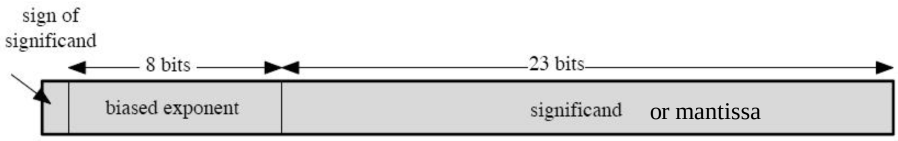
Floating Point Representation



- Significant digits *(Merkitsevät numerot)* and exponent *(suuruusluokka)*
- Normalized number (*Normeerattu muoto*)
  - Most significant digit is nonzero >0
  - Commonly just one digit before the radix point (*desim. pilkku*)

$-0.000\,000\,000\,123 = -1.23 * 10^{-10}$

$0.123 = +1.23 * 10^{-1}$

$123.0 = +1.23 * 10^{2}$

$123\,000\,000\,000\,000 = +1.23 * 10^{14}$

IEEE 754 (floating point) formats

Parameter	Single	Single Extended	Double	Double Extended
Word width (bits)	32	≥ 43	64	≥ 79
Exponent width (bits)	8	≥ 11	11	≥ 15
Exponent bias	127	unspecified	1023	unspecified
Maximum exponent	127	≥ 1023	1023	≥ 16383
Minimum exponent	–126	≤ –1022	–1022	≤ –16382
Number range (base 10)	$10^{-38}, 10^{+38}$	unspecified	$10^{-308}, 10^{+308}$	unspecified
Significand width (bits)*	23	≥ 31	52	≥ 63
Number of exponents	254	unspecified	2046	unspecified
Number of fractions	2^{23}	unspecified	2^{52}	unspecified
Number of values	1.98×2^{31}	unspecified	1.99×2^{63}	unspecified

* not including implied bit

(Sta06 Table 9.3)

32-bit floating point

- 1 b sign
  - 1 = “-”, 0 = “+”
- 8 b exponent
  - Biased representation, no sign (*Ei etumerkkiä, vaan erillinen nollataso*)
    - Exp=5 → store 127+5, Exp=-5 → store 127-5 (bias127)
- 23 b significant (*mantissa*)
  - In normalized form the radix point is preceeded with 1, which is not stored. (hidden bit, Zuse Z3 1939)
- The binary value of the floating point representation

  $-1^{Sign} * 1.Mantissa * 2^{Exponent-127}$

Example

$23.0 = +10111.0 * 2^0 = +1.0111 * 2^4 =?$

127+4=131

sign	exponent	mantissa
0	1000 0011	011 1000 0000 0000 0000 0000

$1.0 = +1.0000 * 2^0 = ?$

0+127 = 127

sign	exponent	mantissa
0	0111 1111	000 0000 0000 0000 0000 0000

Example

0	1000 0000	111 1000 0000 0000 0000 0000
sign	exponent	mantissa

X = ?

$X = (-1)^0 * 1.1111 * 2^{(128-127)}$

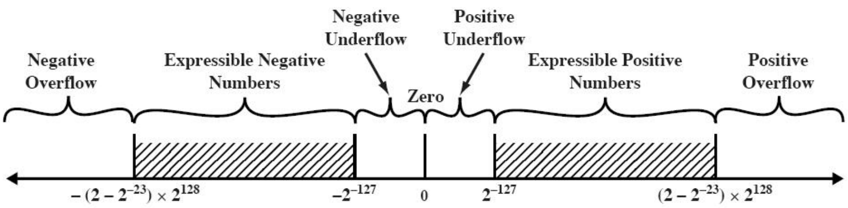
$= 1.1111_2 * 2$

$= (1 + 1/2 + 1/4 + 1/8 + 1/16) * 2$

$= (1 + 0.5 + 0.25 + 0.125 + 0.0625) * 2$

$= 1.9375 * 2$ **$= 3.875$**

Accuracy (*tarkkuus*) (32b)



- Value range (*arvoalue)*
  - 8 b exponent→ $2^{-126} \ldots 2^{127} \sim -10^{-38} \ldots 10^{38}$
- Not exact value
  - 24 b mantissa → $2^{24} \sim 1.7 * 10^{-7} \sim 6$ decimals
- Balancing between range and precision

Numerical errors: Patriot Missile (1991), Ariane 5 (1996)
http://ta.twi.tudelft.nl/nw/users/vuik/wi211/disasters.html

Interpretation of IEEE 754 Floating-Point Numbers

	Single Precision (32 bits)			
	Sign	Biased exponent	Fraction	Value
positive zero	0	0	0	0
negative zero	1	0	0	–0
plus infinity	0	255 (all 1s)	0	∞
minus infinity	1	255 (all 1s)	0	$-\infty$
quiet NaN	0 or 1	255 (all 1s)	$\neq 0$	NaN
signaling NaN	0 or 1	255 (all 1s)	$\neq 0$	NaN
positive normalized nonzero	0	$0 < e < 255$	f	$2^{e-127}(\underline{1.f})$
negative normalized nonzero	1	$0 < e < 255$	f	$-2^{e-127}(1.f)$
positive denormalized	0	0	$f \neq 0$	$2^{e-126}(\underline{0.f})$
negative denormalized	1	0	$f \neq 0$	$-2^{e-126}(0.f)$

Not a Number

Double Precision similarly

(Sta06 Table 9.4)

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NaN: Not a Number

Operation	Quiet NaN Produced by
Any	Any operation on a signaling NaN
Add or subtract	Magnitude subtraction of infinities: $(+\infty) + (-\infty)$ $(-\infty) + (+\infty)$ $(+\infty) - (+\infty)$ $(-\infty) - (-\infty)$
Multiply	$0 \times \infty$
Division	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
Remainder	x REM 0 or ∞ REM y
Square root	$\sqrt{x}$ where $x < 0$

(Sta06 Table 9.6)

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Floating Point Arithmetics (*Liukulukuaritmetiikkaa*)

IEEE-754 Standard
Addition
Subtraction
Multiplication
Division

Floating point arithmetics

- Calculations need wide registers
  - Guard bits - pad right end of significand
  - More bits for the significand (mantissa)
  - Using Denormalized formats
- Addition and subtraction
  - More complex than multiplication
  - Operands must have same exponent
    - Denormalize the smaller operand (alignment!)
      - Loss of digits (less precise and missing information)
  - Result (must) be normalised
- Multiplication and division
  - Significand and exponent handled separately

Floating point arithmetics

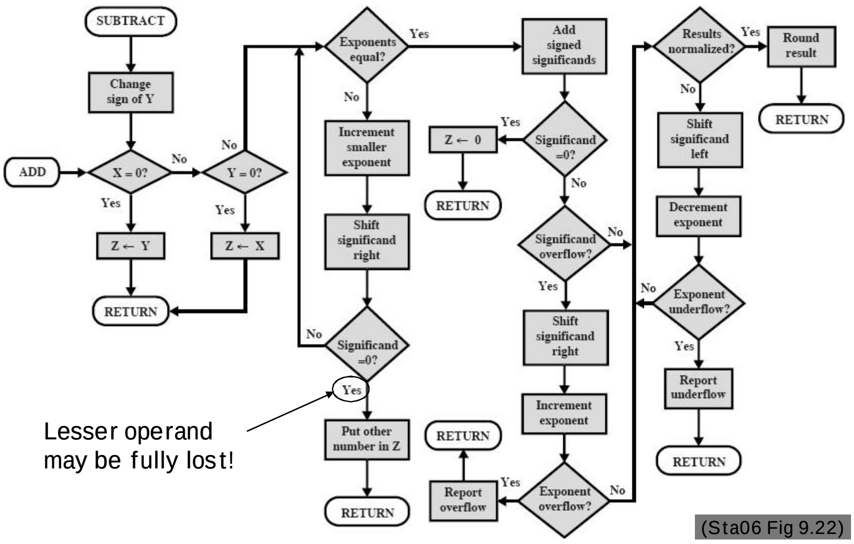
Floating Point Numbers	Arithmetic Operations
$X = X_S \times B^{X_E}$ $Y = Y_S \times B^{Y_E}$	$\left.\begin{array}{l} X + Y = \left(X_s \times B^{X_E - Y_E} + Y_s\right) \times B^{Y_E} \\ X - Y = \left(X_s \times B^{X_E - Y_E} - Y_s\right) \times B^{Y_E} \end{array}\right\} X_E \le Y_E$ $X \times Y = \left(X_s \times Y_s\right) \times B^{X_E + Y_E}$ $\frac{X}{Y} = \left(\frac{X_s}{Y_s}\right) \times B^{X_E - Y_E}$

(Sta06 Table 9.5)

$X = 0.3 \times 10^2 = 30$
$Y = 0.2 \times 10^3 = 200$

$X + Y = (0.3 \times 10^{2-3} + 0.2) \times 10^3 = 0.23 \times 10^3 = 230$
$X - Y = (0.3 \times 10^{2-3} - 0.2) \times 10^3 = (-0.17) \times 10^3 = -170$
$X \times Y = (0.3 \times 0.2) \times 10^{2+3} = 0.06 \times 10^5 = 6000$
$X \div Y = (0.3 \div 0.2) \times 10^{2-3} = 1.5 \times 10^{-1} = 0.15$

Addition and Subtraction



(Sta06 Fig 9.22)

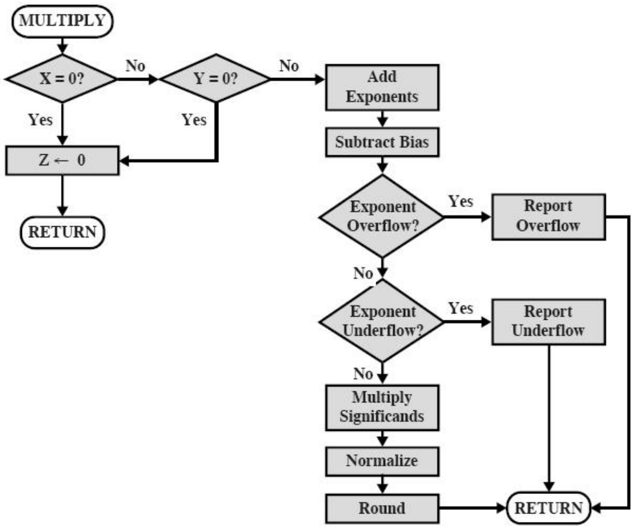
Special cases

- Exponent overflow (*eksponentin ylivuoto*)
  - Very large number (above max) — Programmable option
  - Value ∞ or -∞ , alternatively cause exception
- Exponent underflow (*eksponentin alivuoto*)
  - Very small number (below min) — Programmable option
  - Value 0 (or cause exception)
- Significand overflow (*mantissan ylivuoto*)
  - Normalise! — Fix it!
- Significand underflow (*mantissan alivuoto*)
  - Denormalizing may lose the significand accuracy
  - All significant bits lost? — Ooops, lost data!

Rounding (pyöristys)

- Example — 3.1234, -4.5678
  - Value has four decimals
  - Present it using only 3 decimals
  - Normal rounding rule — 3.123, -4.568
    round to nearest value
  - Always towards ∞ (*ylöspäin*) — 3.124, -4.567
  - Always towards -∞ (*alaspäin*) — 3.123, -4.568
  - Always towards 0 — 3.123, -4.567
- For example, Intel Itanium supports all of these alternatives

Multiplication



(Sta06 Fig 9.23)

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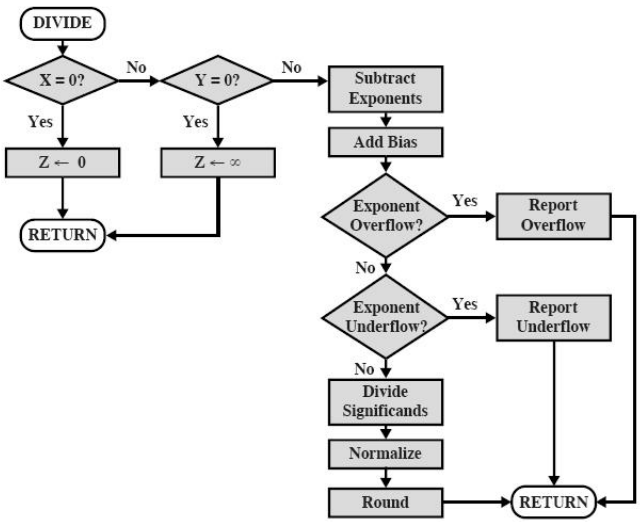
Division



(Sta06 Fig 9.24)

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Review Questions / Kertauskysymyksiä

- Why we use twos complement?
- How does twos complement "expand" to a large number of bits (8b →16 b)?
- Format of single-precision floating point number?
- When does underflow happen?

- Miksi käytetään 2:n komplementtimuotoa?
- Miten 2:n komplementtiesitys laajenee "suurempaan tilaan" (esim. 8b esitys →16 b:n esitys)?
- Millainen on yksinkertaisen tarkkuuden liukuluvun esitysmuoto?
- Milloin tulee liukuluvun alivuoto?

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