HELSINGIN YLIOPISTO
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Digital logic

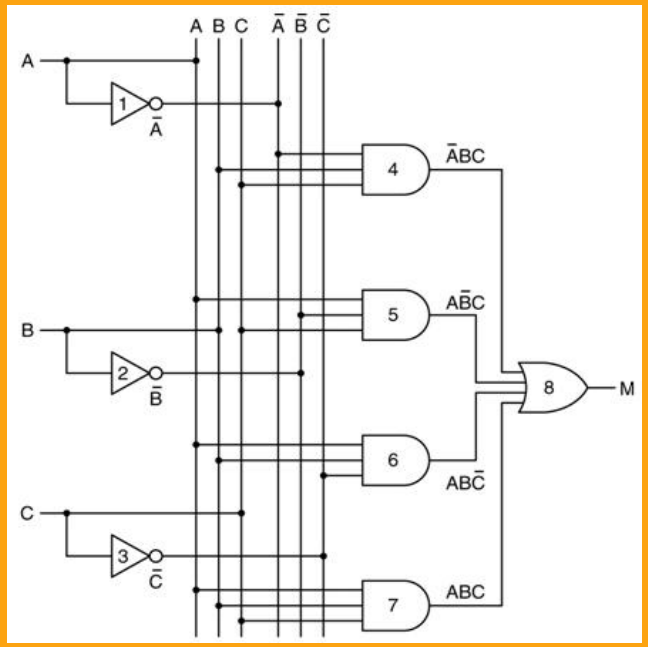
Stallings: Appendix B

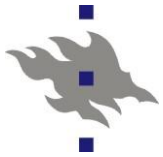
Boolean Algebra

Combinational Circuits

Simplification

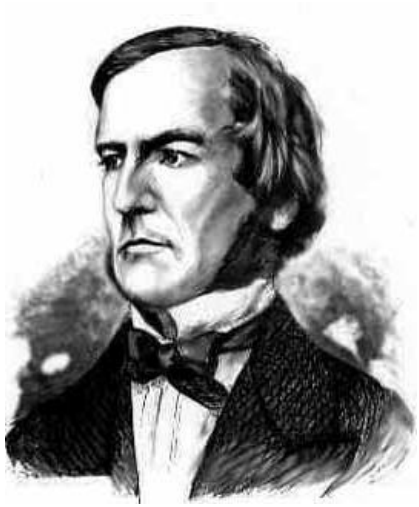
Sequential Circuits





Boolean Algebra

Boolean Algebra

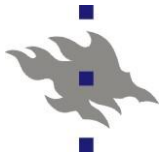


- George Boole
  - ideas 1854
- Claude Shannon (kuva) (gradu)
  - apply to circuit design, 1938
  - “father of information theory”

Topics:

- Describe digital circuitry function
  - programming language?
- Optimise given circuitry
  - use algebra (Boolean algebra) to manipulate (Boolean) expressions into simpler expressions

(piirisuunnittelu)



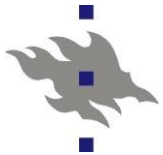
Boolean Algebra

- Variables: A, B, C
- Values: TRUE (1), FALSE (0)
- Basic logical operations:
  - binary: AND (·) $A \bullet B = AB$ — ja — product
    OR (+) $B + C$ — tai — sum
  - unary: NOT (¯) $\overline{A}$ — ei — negation

integer arithmetics

- Composite operations, equations
  - precedence: NOT, AND, OR
  - parenthesis

$$D = A + \overline{B} \bullet C = A + ((\overline{B})C)$$



Boolean Algebra

- Other operations
  - XOR (exclusive-or)
  - NAND
  - NOR

$$A \text{ NAND } B = \text{NOT}(A \text{ AND } B) = \overline{AB}$$

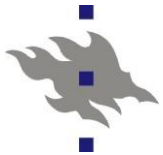
$$A \text{ NOR } B = \text{NOT}(A \text{ OR } B) = \overline{A + B}$$

- Truth tables

Boolean Operators

P	Q	NOT P	P AND Q	P OR Q	P XOR Q	P NAND Q	P NOR Q
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	1	0
1	1	0	1	1	0	0	0

(Sta06 Table B.1)

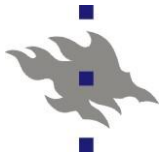


Postulates and Identities

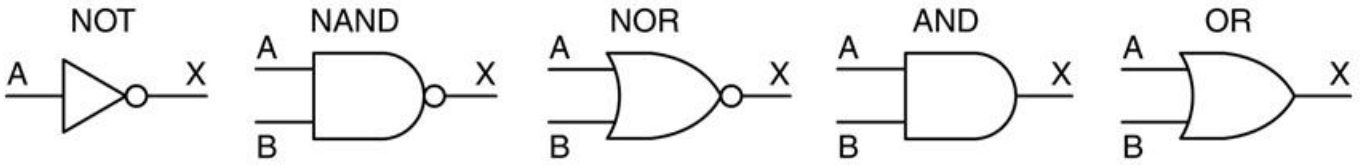
- How can I manipulate expressions?
  - Simple set of rules?

Basic Postulates		
$A \cdot B = B \cdot A$	$A + B = B + A$	Commutative Laws vaihdantalaki
$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$	$A + (B \cdot C) = (A + B) \cdot (A + C)$	Distributive Laws osittelulaki
$1 \cdot A = A$	$0 + A = A$	Identity Elements neutraalialkiot
$A \cdot \overline{A} = 0$	$A + \overline{A} = 1$	Inverse Elements alkion ja komplementin tulo ja summa
Other Identities		
$0 \cdot A = 0$	$1 + A = 1$	tulo 0'n kanssa, summa 1'n kanssa
$A \cdot A = A$	$A + A = A$	tulo ja summa itsensä kanssa
$A \cdot (B \cdot C) = (A \cdot B) \cdot C$	$A + (B + C) = (A + B) + C$	Associative Laws liitäntälait
$\overline{A \quad B} = \overline{A} + \overline{B}$	$\overline{A + B} = \overline{A} \quad \overline{B}$	DeMorgan's Theorem

(Sta06 Table B.2)



Gates (veräjät / portit)

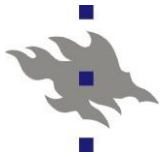


- Implement basic Boolean algebra operations
- Fundamental building blocks
  - 1 or 2 inputs, 1 output
- Combine to build more complex circuits
  - memory, adder, multiplier, …
- Gate delay
  - change inputs, after gate delay new output available
  - 1 ns? 10 ns? 0.1 ns?

yhteenlaskupiiri,
kertolaskupiiri

http://tech-www.informatik.uni-hamburg.de/applets/cmos/cmosdemo.html (extra material)

Sta06 Fig B.1



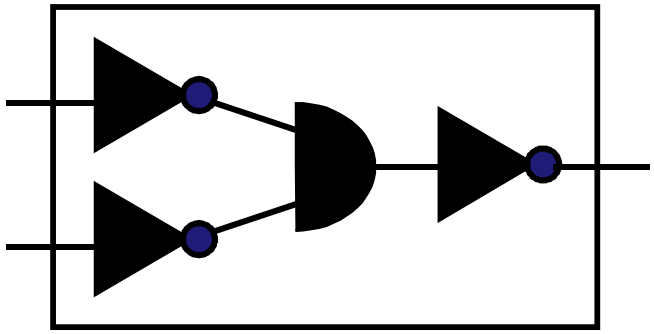
Functionally Complete Set

funktionaalisesti täydellinen joukko => joukosta voidaan muodostaa kaikki portit

- Can build all basic gates (AND, OR, NOT) from a smaller set of gates
  - With AND, NOT
  - With OR, NOT
  - With NAND alone
  - With NOR alone

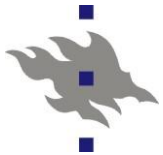
(Nämä seuraavat suoraan DeMorganin kaavoista)

$$A + B = \overline{\overline{A} \bullet \overline{B}}$$



OR with AND and NOT gates

Sta06 Fig B.2, B.3

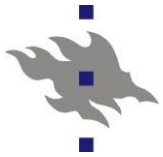


Combinational Circuits

yhdistelmäpiirit

Sta06 Fig B.4

- Interconnected set of gates
  - m inputs, n outputs
  - change inputs, wait for gate delays, new outputs
- Each output
  - depends on combination of input signals
  - can be expressed as Boolean function of inputs
- Function can be described in three ways
  - with Boolean equations (one equation for each output)
  - with truth table
  - with graphical symbols for gates and wires



Describing the Circuit

Boolean equations

$$F = \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$

Truth table

<---------- inputs ----------> <- output ->

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

(Sta06 Table B.3)

Graphical symbols

Sta06 Fig B.4

Simplification

Piirin yksinkertaistaminen

Simplify Presentation (and Implementation)

- Boolean equations
  - Sum of products form (SOP)

Sta06 Table B.3

tulojen summa

Sta06 Fig B.4

$$F = \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$

Boolean algebra

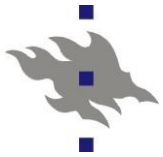
  - Product of sums form (POS)

summien tulo

$$F = (A + B + C) \bullet (A + B + \overline{C}) \bullet (\overline{A} + B + C) \bullet (\overline{A} + B + \overline{C}) \bullet (\overline{A} + \overline{B} + \overline{C})$$

Sta06 Fig B.5

- Which presentation is better?
  - Fewer gates? Smaller area on chip?
  - Smaller circuit delay? Faster?



Algebraic Simplification

- Circuits become too large to handle?
- Use basic identities to simplify Boolean expressions

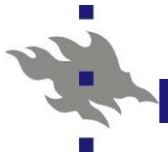
$$F = \overline{A}B\overline{C} + \overline{A}BC + A\overline{B}\overline{C}$$
$$= \overline{A}B + B\overline{C} = B(\overline{A} + \overline{C})$$

Sta06 Fig B.4

Sta06 Fig B.6

- May be difficult to do!
- How to do it automatically?
- Build a program to do it “best”?

$$f = \overline{a}\overline{b}cd + \overline{a}bcd + ab\overline{c}\overline{d} + ab\overline{c}d$$
$$+ abcd + abc\overline{d} + a\overline{b}cd + a\overline{b}c\overline{d}$$



How so?

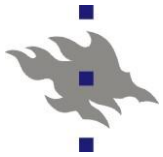
$$F = \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$
$$= \overline{A}B\overline{C} + \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C}$$
$$= (\overline{A}B\overline{C} + \overline{A}BC) + (\overline{A}B\overline{C} + AB\overline{C})$$
$$= \overline{A}B(\overline{C} + C) + (\overline{A} + A)B\overline{C}$$
$$= \overline{A}B(1) + (1)B\overline{C}$$
$$= \overline{A}B + B\overline{C}$$
$$= B(\overline{A} + \overline{C})$$

Boolean algebra:
A+A =A

And this?

Entäs tämä?

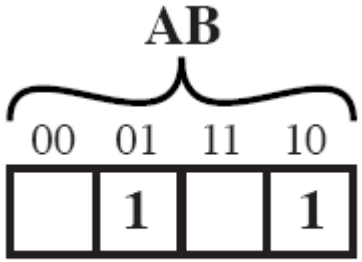
$$f = \overline{a}\overline{b}cd + \overline{a}bcd + ab\overline{c}\overline{d} + ab\overline{c}d$$
$$+ abcd + abc\overline{d} + a\overline{b}cd + a\overline{b}c\overline{d}$$



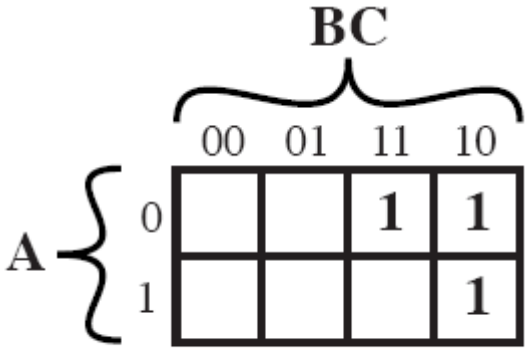
Karnaugh Map

Karnaugh kartta

- Represent Boolean function (i.e., circuit) truth table in another way
  - Use canonical form: each term has each variable once
  - Use SOP presentation
- Karnaugh map squares
  - Each square is one product (input value combination)
  - Value is one (1) iff the product is present
    o/w value is “empty”

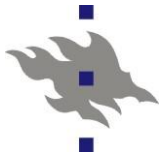


(a) $F = A\bar{B} + \bar{A}B$



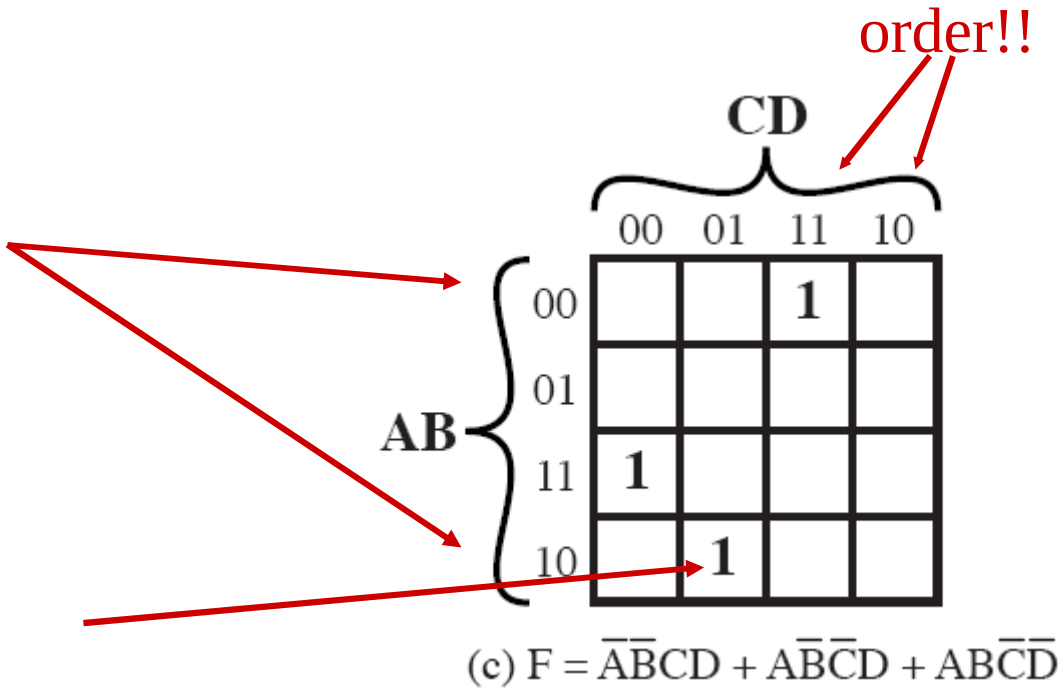
(b) $F = \bar{A}B\bar{C} + \bar{A}BC + AB\bar{C}$

(Sta06 Fig B.7)



Karnaugh Map

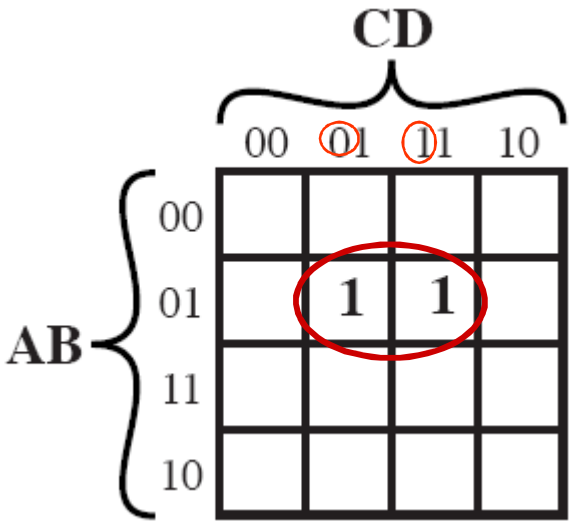
- Adjacent squares differ only in one input value (wrap around)
- Square for input combination $A\overline{B}\overline{C}D$ 1001



(Sta06 Fig B.7)

Karnaugh Map Simplification

- If adjacent squares have value 1, input values differ only in one variable
- Value of that variable is irrelevant (when all other input variables are fixed for those squares)



- Can ignore that variable for those expressions
  - + $\overline{A}B\overline{C}D + \overline{A}BC\overline{D} +$ → ignore C $+ \overline{A}B\overline{D} +$

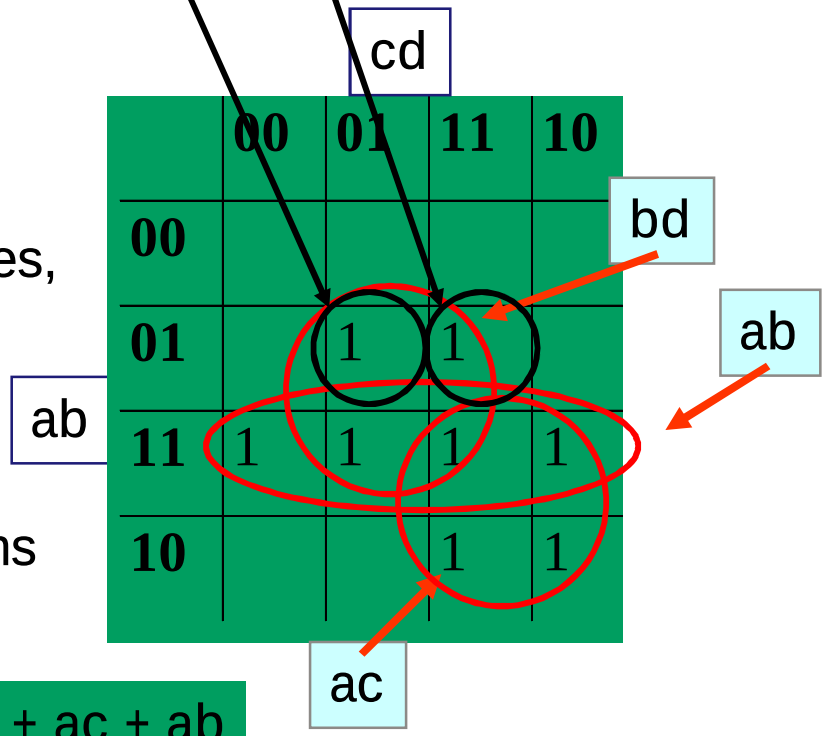
Using Karnaugh Maps to Minimize Boolean Functions (8)

Original function

$$f = \bar{a}\bar{b}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}d + \bar{a}\bar{b}c\bar{d} + \bar{a}\bar{b}cd + \bar{a}b\bar{c}\bar{d} + \bar{a}bc\bar{d} + \bar{a}bcd + ab\bar{c}\bar{d}$$

Canonical form (already OK)

Karnaugh Map

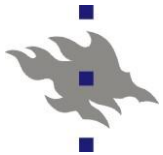


Find smallest number of circles, each with largest number (2^i) of 1's

- can wrap-around

Select parameter combinations corresponding to the circles

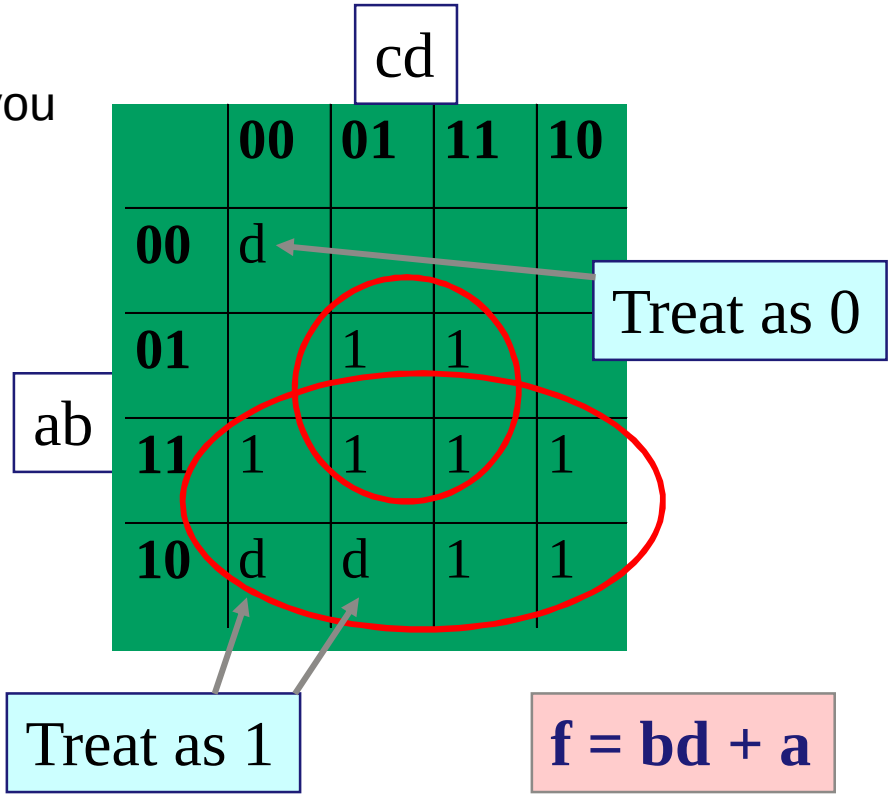
Get reduced function f = bd + ac + ab



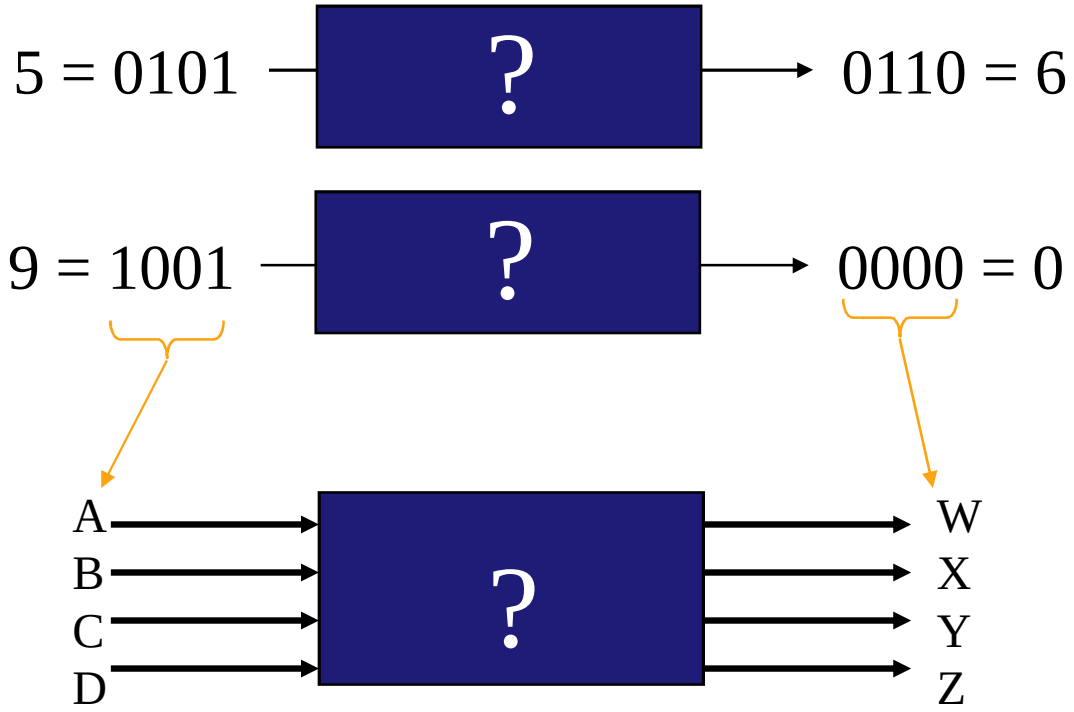
Impossible Input Variable Combinations

(3)

- What if some input combinations can never occur?
  - Mark them ”don’t care”, “d”
  - Treat them as 0 or 1, whichever is best for you
  - More room to optimize



Example: Circuit to add 1 (mod 10) to 4-bit BCD decimal number (3)



- Truth table?
- Karnaugh maps for W, X, Y and Z?

Example cont.: Truth Table

Truth Table for the One-Digit Packed Decimal Incrementer

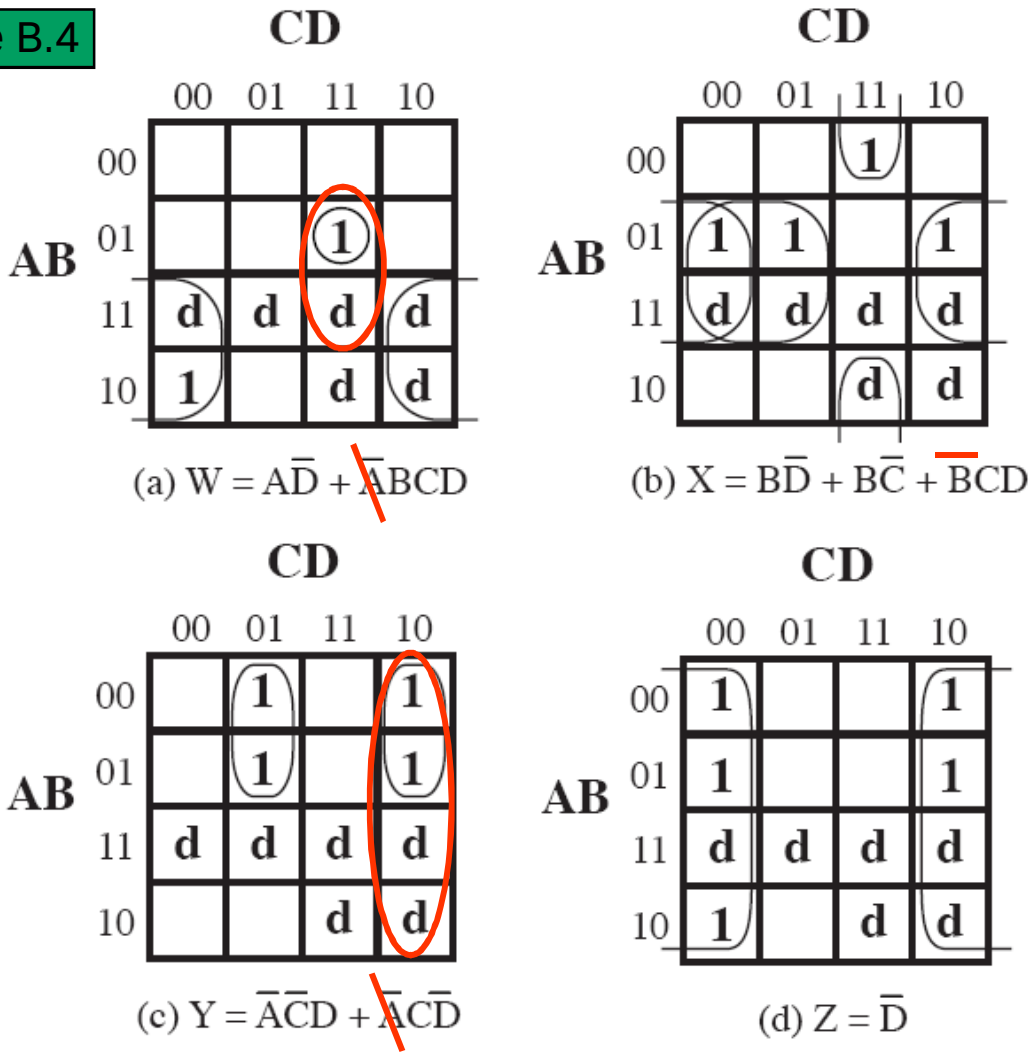
	Input					Output			
Number	A	B	C	D	Number	W	X	Y	Z
0	0	0	0	0	1	0	0	0	1
1	0	0	0	1	2	0	0	1	0
2	0	0	1	0	3	0	0	1	1
3	0	0	1	1	4	0	1	0	0
4	0	1	0	0	5	0	1	0	1
5	0	1	0	1	6	0	1	1	0
6	0	1	1	0	7	0	1	1	1
7	0	1	1	1	8	1	0	0	0
8	1	0	0	0	9	1	0	0	1
9	1	0	0	1	0	0	0	0	0
Don't care condition	1	0	1	0		d	d	d	d
	1	0	1	1		d	d	d	d
	1	1	0	0		d	d	d	d
	1	1	0	1		d	d	d	d
	1	1	1	0		d	d	d	d
	1	1	1	1		d	d	d	d

No carry!

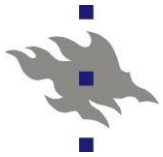
(Sta06 Table B.4)

Example cont: Karnaugh Map

Sta06 Table B.4

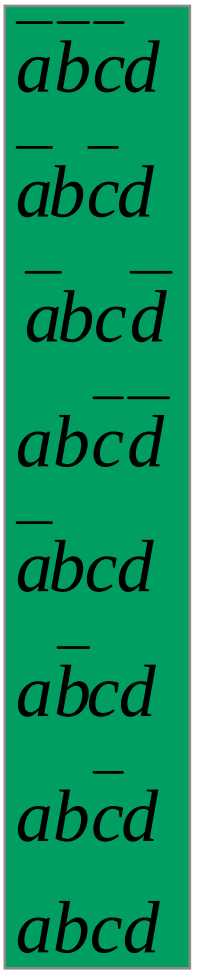


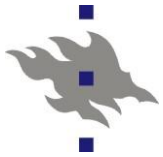
(Sta06 Fig B.10)



Other Methods to simplify Boolean expressions

- Why?
  - Karnaugh maps become complex with
  - 6 input variables
- Quine-McKluskey method
  - Tabular method
  - Automatically suitable for programming
- Luque Method
  - Based on dividing circle in different ways
  - Can be fractally expanded to infinitely many variables
- Interesting, but not part of this course
- Details skipped





Basic

Combinatorial Circuits

Building blocks for more complex circuits

- Multiplexer
- Encoders/decoder
- Read-Only-Memory
- Adder

limitin

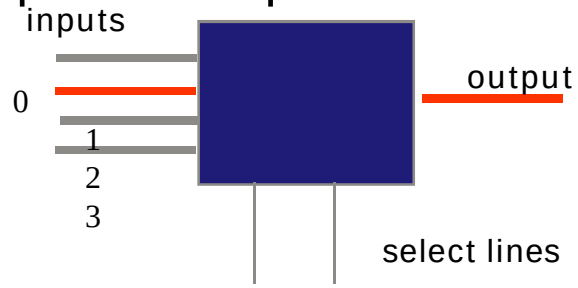
Multiplexers

Sta06 Fig B.12

- Select one of many possible inputs to output
  - black box
  - truth table
  - implementation

Sta06 Table B.7

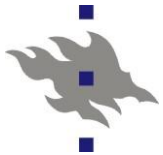
Sta06 Fig B.13



- Each input/output “line” can be many parallel lines
  - select one of three 16 bit values
    - $C_{0..15}$, $IR_{0..15}$, $ALU_{0..15}$
  - simple extension to one line selection
    - lots of wires, plenty of gates …

Sta06 Fig B.14

- Used to control signal and data routing
  - Example: loading the value of PC



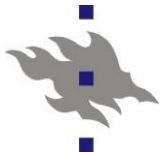
Encoders/Decoders

- Exactly one of many Encoder input or Decoder output lines is 1
- Encode that line number as output
  - hopefully less pins (wires) needed this way
  - optimise for space, not for time
  - Example:
    - encode 8 input wires with 3 output pins
    - route 3 wires around the board
    - decode 3 wires back to 8 wires at target

space-time tradeoff

Sta06 Fig B.15

Ex. Choosing the right memory chip from the address bits.



Read-Only-Memory (ROM) (5)

- Given input values, get output value
  - Like multiplexer, but with **fixed data**
- Consider input as address, output as contents of memory location

- Example
  - Truth tables for a ROM
    - 64 bit ROM
    - 16 words, each 4 bits wide

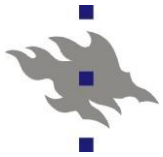
Sta06 Table B.8

Mem (7) = 4

Mem (11) = 14

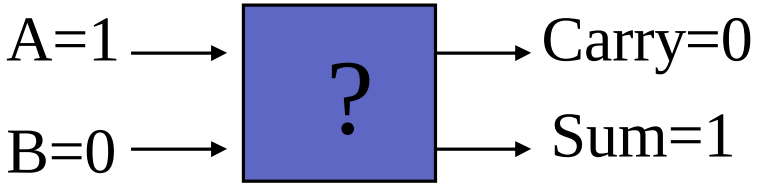
  - Implementation with decoder & or gates

Sta06 Fig B.20

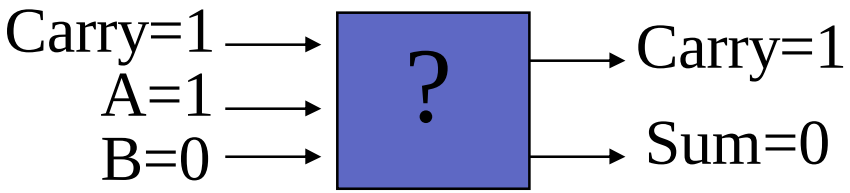


Adders

- 1-bit adder



- 1-bit adder with carry



- Implementation

Sta06 Table B.9, Fig B.22

Compare to ROM?

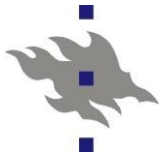
- Build a 4-bit adder from four 1-bit adders

Sta06 Fig B.21

Sequential Circuits

sarjalliset piirit

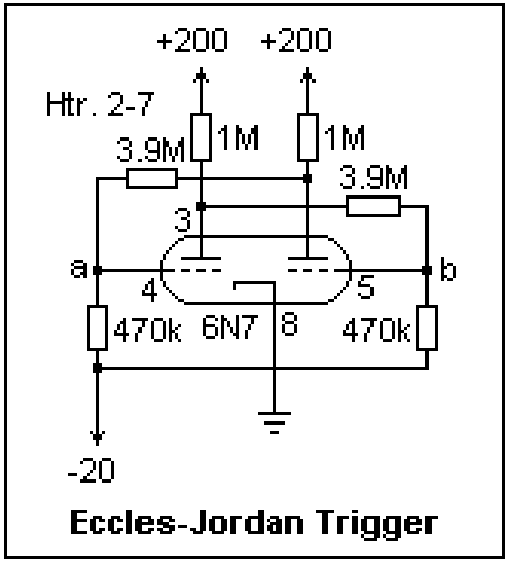
- Flip-Flop
- S-R Latch
- Registers
- Counters



Sequential Circuit (sarjallinen piiri)

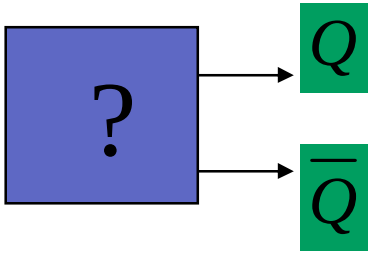
- Circuit has (modifiable) internal state
  - remembers its previous state
- Output of circuit depends (also) on internal state
  - not only from current inputs
  - output = f_o(input, state)
  - new state = f_s(input, state)
- Circuits needed for
  - processor control
  - registers
  - memory

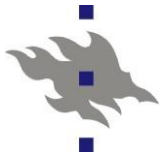
http://www.du.edu/~etuttle/electron/elect36.htm



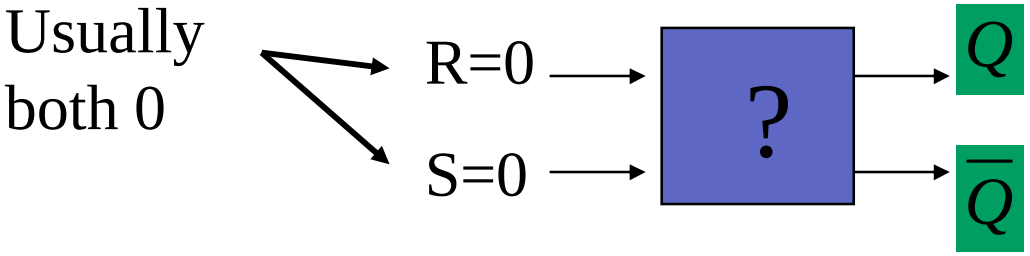
Flip-Flop (kiikku)

- William Eccles & F.W. Jordan
  - with vacuum tubes, 1919
- 2 states for Q (0 or 1, true or false)
- 2 outputs
  - complement values
  - both always available on different pins
- Need to be able to change the state (Q)



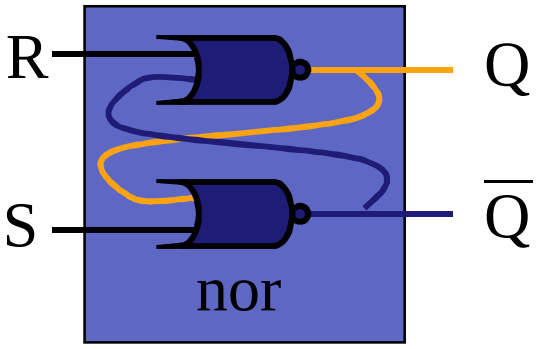


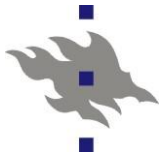
S-R Flip-Flop or S-R Latch (salpa)



S = “SET” = “Write 1” = “set S=1 for a short time”
R = “RESET” = “Write 0” = “set R=1 for a short time”

nor (0, 0) = 1
nor (0, 1) = 0
nor (1, 0) = 0
nor (1, 1) = 0



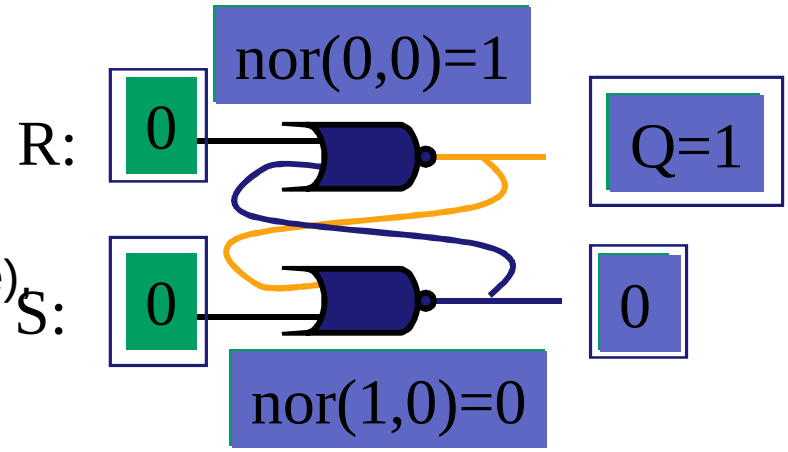


S-R Latch Stable States (4)

- 1 bit memory (value = value of Q)
- bistable, when R=S=0
  - Q=0?
  - Q=1?

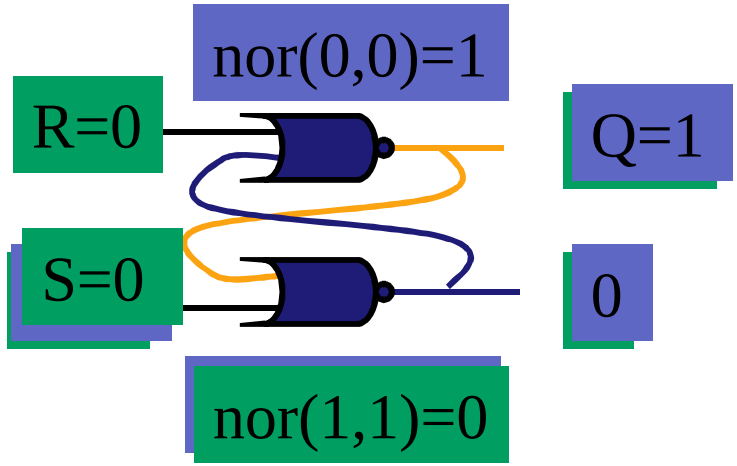
nor (0, 0) = 1
nor (0, 1) = 0
nor (1, 0) = 0
nor (1, 1) = 0

$t = f_o(input, state)$
$= f_s(input, state)$



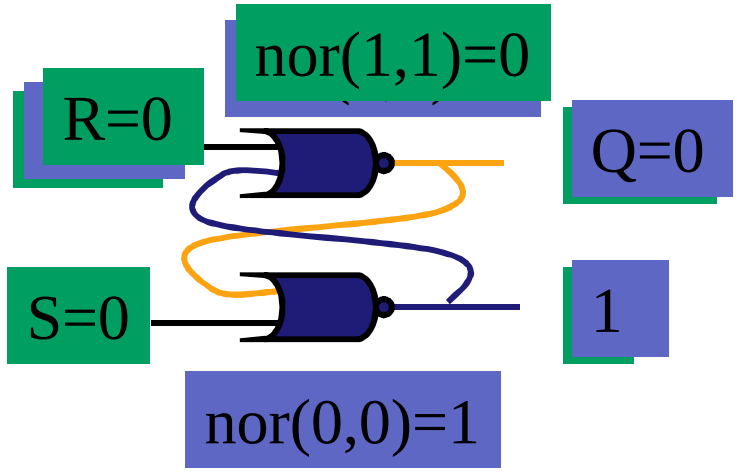
S-R Latch Set (=1) and Reset (=0) (17)

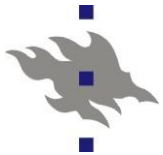
Write 1: S= $0 \rightarrow 1 \rightarrow 0$



Write 0: R= $0 \rightarrow 1 \rightarrow 0$

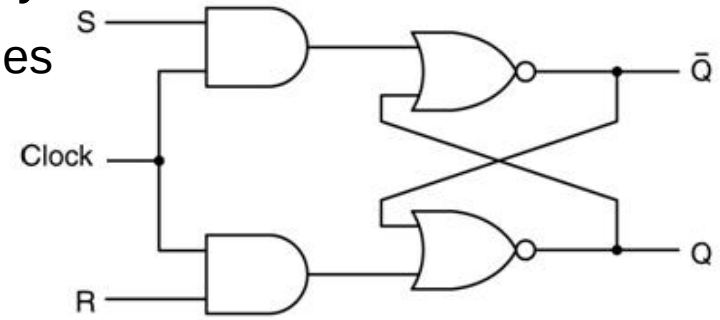
nor (0, 0) = 1
nor (0, 1) = 0
nor (1, 0) = 0
nor (1, 1) = 0





Clocked Flip-Flops

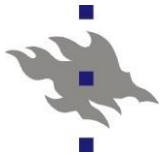
- State change can happen only when clock is 1
  - more control on state changes



(Sta06 Fig B.26)

- Clocked S-R Flip-Flop
- D Flip-Flop Sta06 Fig B.27
  - only one input D
    - D = 1 and CLOCK → write 1
    - D = 0 and CLOCK → write 0
- J-K Flip-Flop Sta06 Fig B.28
  - Toggle Q when J=K=1

Sta06 Fig B.29

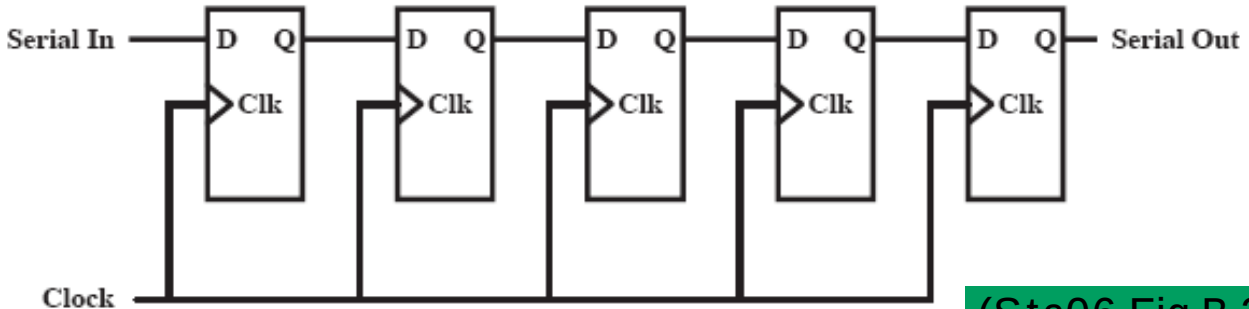


Registers

- Parallel registers
  - read/write
  - CPU user registers
  - additional internal registers

Sta06 Fig B.30

- Shift Registers
  - shifts data 1 bit to the right
  - serial to parallel?
  - ALU ops?
  - rotate?



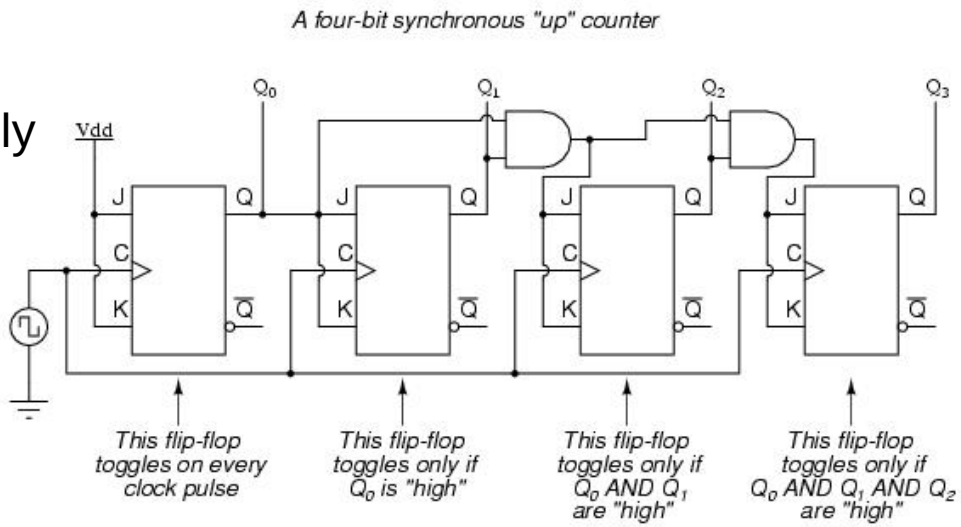
(Sta06 Fig B.31)

Counters

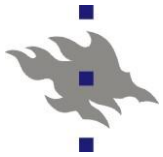
- Add 1 to stored counter value
- Counter
  - parallel register plus increment circuits
- Ripple counter (aalto, viive)
  - asynchronous
  - increment least significant bit,
  - and handle “carry” bit as far as needed
- Synchronous counter
  - modify all counter flip-flops simultaneously
  - faster, more complex, more expensive

Sta06 Fig B.32

space-time tradeoff



(http://www.allaboutcircuits.com)



Summary

- Boolean Algebra → Gates → Circuits
  - can implement all with NANDs or NORs
  - simplify circuits:
    - Karnaugh, (Quine-McKluskey, Luque, …)

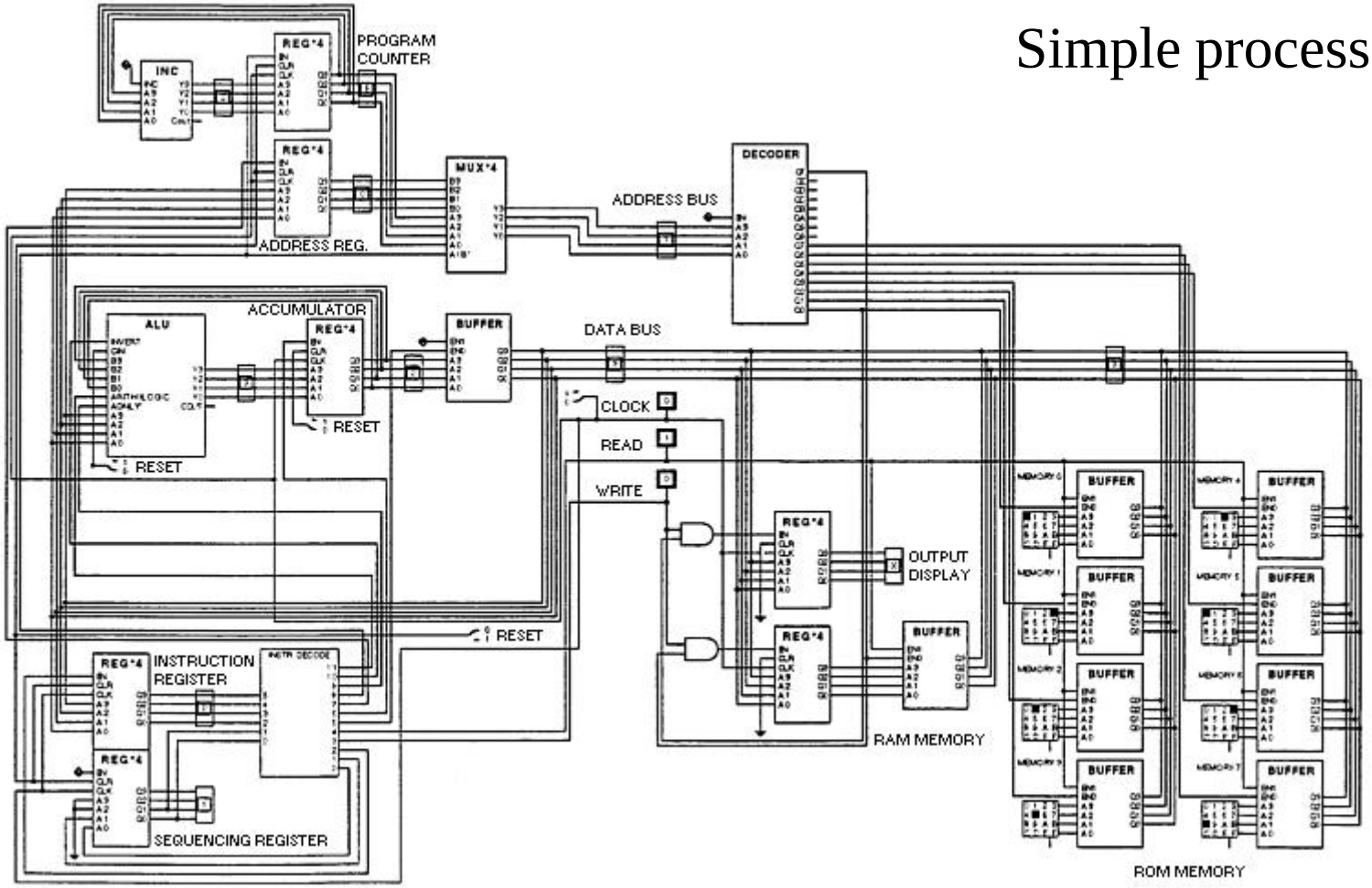
- Components for CPU design
  - ROM, adder
  - multiplexer, encoder/decoder
  - flip-flop, register, shift register, counter

Simulations of gates and circuits:

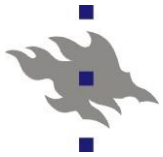
Hades Simulation Framework: http://tams-www.informatik.uni-hamburg.de/applets/hades/webdemos/index.html

-- End of Appendix B: Digital Logic --

Simple processor



http://www.gamezero.com/team-0/articles/math_magic/micro/stage4.html



Kertauskysymyksiä/Review questions

- DeMorganin laki?
- Miten boolen funktio minimoidaan Karnaugh- kartan avulla?
- Mitä eroa sarjallisessa piirissä on verrattuna ”normaaliin” kombinatoriseen piiriin?
- Miten S-R kiikku toimii?

- DeMorgan’s theorem?
- How to minimize a Boolean function using Karnaughs map?
- How do sequential circuits differ from ’normal’ combinational circuits?
- How does the S-R flip-flop function?