Lecture 5

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Computer Arithmetic

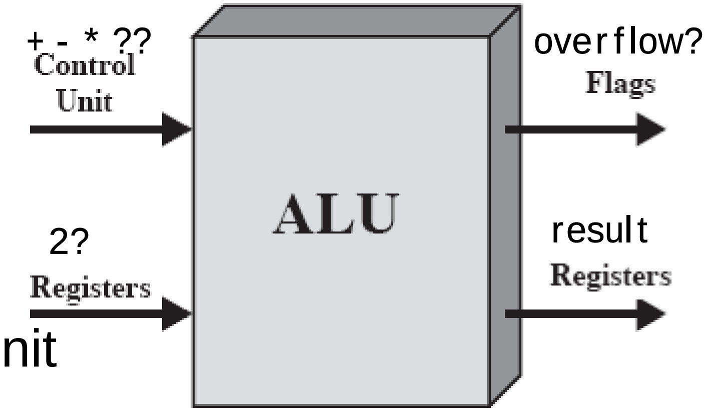
Ch 9 [Stal10]

Integer arithmetic

Floating-point arithmetic

ALU

- ALU = Arithmetic Logic Unit (*Aritmeettis-looginen yksikkö*)
- Actually performs operations on data
  - Integer and floating-point arithmetic
  - Comparisons (*vertailut*), left and right shifts (*sivuttaissiirrot*)
  - Copy bits from one register to another
  - Address calculations (*Osoitelaskenta*): branch and jump (*hypyt*), memory references (*muistiviittaukset)*
- Data from/to internal registers (latches)
  - Input copied from normal registers (or from memory)
  - Output goes to register (or memory)
- Operation
  - Based on instruction register, control unit



(Sta10 Fig 9.1)

Integer Representation (*kokonaislukuesitys*)

- Binary representation, bit sequence, only 0 and 1
- ”Weight” of the digit based on position

$57 = 5*10^1 + 7*10^0$

$= 32 + 16 + 8 + 1$

$= 1*2^5 + 1*2^4 + 1*2^3 + 0*2^2 + 0*2^1 + 1*2^0$

$= 0011\ 1001$

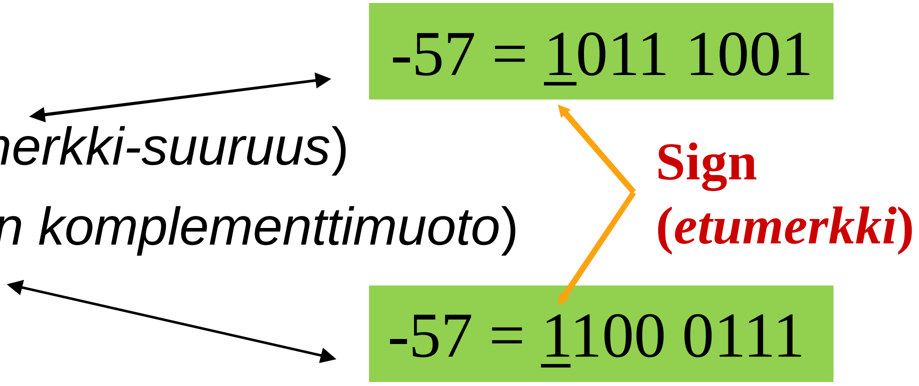
$= \underline{0x}39$ (hexadecimal)

$= 3*16^1 + 9*16^0$

- Most significant bit, MSB (*eniten merkitsevä bitti*)
- Least significant bit, LSB (*vähiten merkitsevä bitti*)

Integer Representation

- Negative numbers?
  - Sign magnitude (*Etumerkki-suuruus*)
  - Twos complement (*2:n komplementtimuoto*)



$-57 = \underline{1}011\ 1001$

Sign
(*etumerkki*)

$-57 = \underline{1}100\ 0111$

- Computers use twos complement
  - Just one zero (no +0 and -0)
    - Comparison to zero easy
  - Math is easy to implement
    - No need to consider sign
    - Subtraction becomes addition
  - Simple hardware and circuit

+2 = 0000 0010
+1 = 0000 0001
0 = 0000 0000
-1 = 1111 1111
-2 = 1111 1110

Twos complement (*2:n komplementti*)

- Example
  - 8-bit sequence, value -57

57 = 0011 1001	unsigned value (*itseisarvo*)
1100 0110	invert bits (ones complement)
1100 0110	
$\underline{1}$	add 1
$\underline{1}100\ 0111$	twos complement

Reject overflow

  - Easy to expand. As a 16-bit sequence

$57 = \underline{0}011\ 1001 = \underline{0000\ 0000}\ \underline{0}011\ 1001$

$-57 = \underline{1}100\ 0111 = \underline{1111\ 1111}\ \underline{1}100\ 0111$

sign extension

Twos Complement Addition

- Twos complement value range (*arvoalue*): $-2^{n-1} \ldots 2^{n-1} -1$

  8 bits: $-2^7 \ldots 2^7-1 = -128 \ldots 127$

  32 bits: $-2^{31} \ldots 2^{31}-1 = -2\,147\,483\,648 \ldots 2\,147\,483\,647$

- Addition overflow (*ylivuoto*) easy to detect
  - No overflow, if different signs in operands
  - Overflow, if same sign (*etumerkki*)

    and the results sign differs from the operands

How would you implement this with and/or gates?

57 = 0011 1001

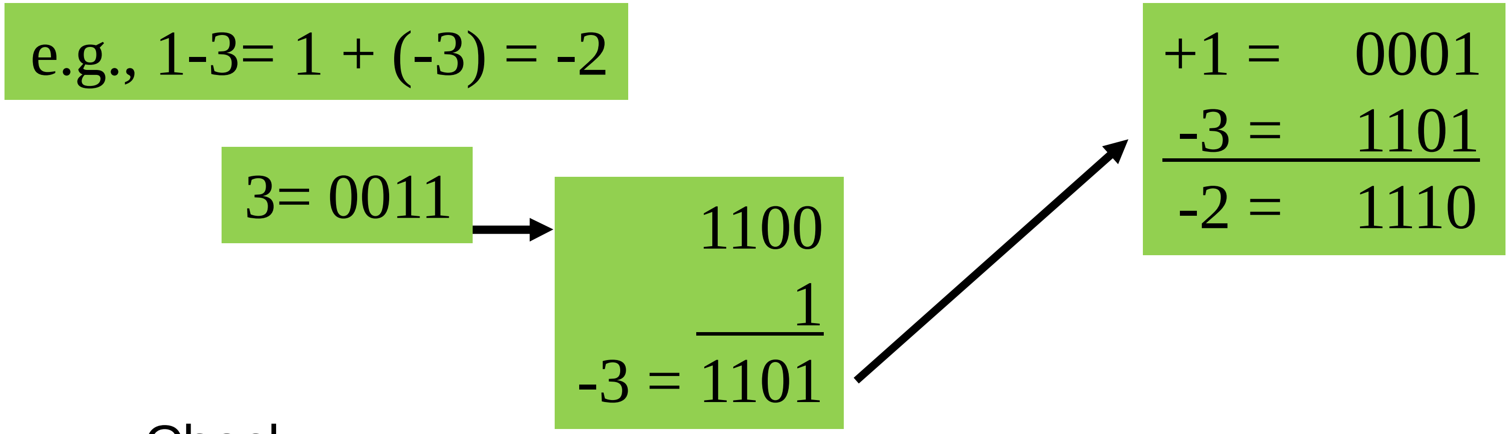
+ 80 = 0101 0000

---

137 = $\underline{1}$000 1001 Overflow!

Twos Complement Subtraction

- Subtraction as addition
  - Forget the sign, handle as if unsigned!
  - Complement 2nd term, the subtrahend, then add (*lisää 2:n komplementti vähentäjästä*)
  - Simple hardware



e.g., 1-3= 1 + (-3) = -2

3= 0011

1100
1
-3 = 1101

+1 = 0001
-3 = 1101
-2 = 1110

- Check
  - Overflow? (same rule as in addition)
  - sign= 1, result is negative

Twos Complement Negation

- 1: invert all bits
- 2: add 1
- 3: Special cases
  - Ignore carry bit (*ylivuotobitti*)
  - Sign really changed?
    - Cannot negate smallest negative
    - Result in exception
- Simple hardware

-57 = 1100 0111
0011 1000
1
+57 = 0011 1001

-128 = 1000 0000
0111 1111
1
1000 0000

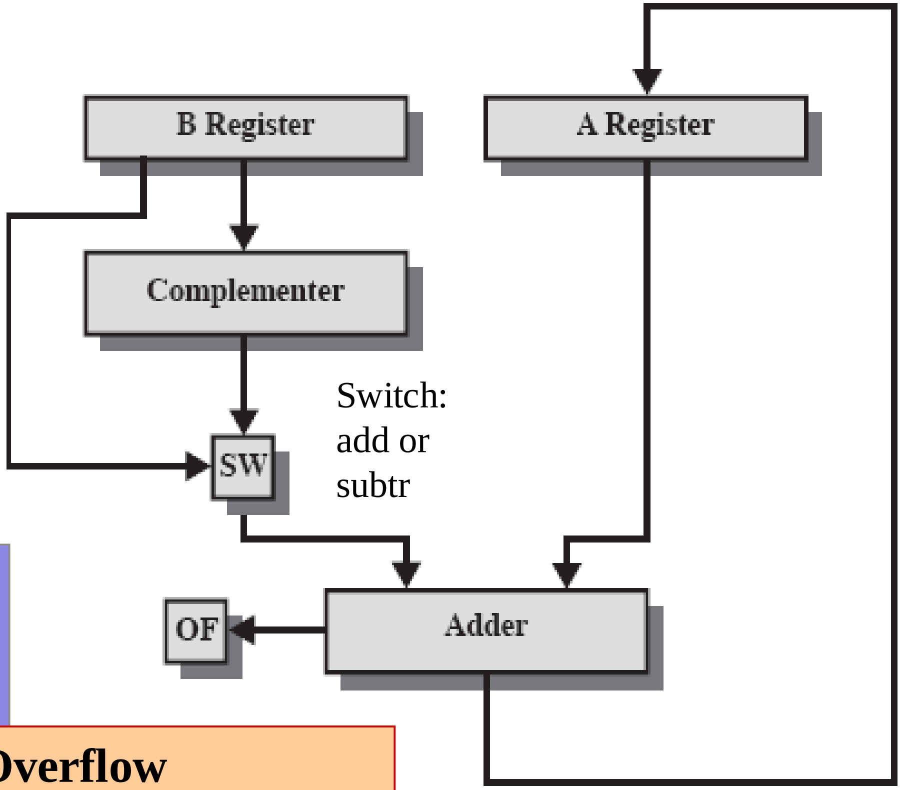
Integer Addition (and Subtraction)

- Normal binary addition
  - In subtraction: complement the 2. operand, subtrahend (*vähentäjä*) and add to 1. operand, minuend (*vähennettävä*)
- Ignore carry
  - Check sign for Overflow indication
- Simple hardware function
  - Two circuits:

    Complement and addition

-4-1=?	-4-5=?
1100 =-4	1100 = -4
+1111 =-1	+1011 = -5
11011 =-5	10111 = ?

Overflow
(4-bit 2-compl ints)

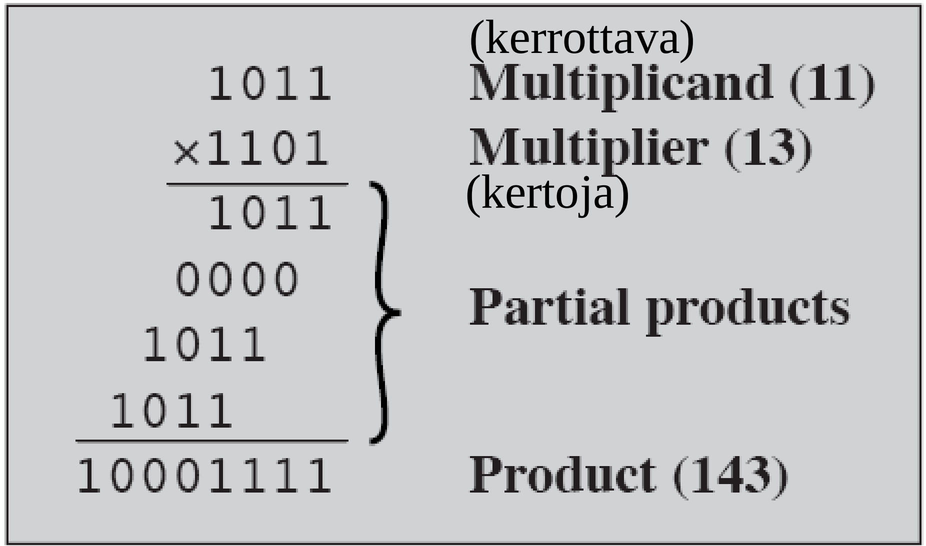


Switch:
add or
subtr

(Sta10 Fig 9.6)

Integer Multiplication

- ”Just like” you learned at school
  - Easy with just 0 and 1!
- Hardware?
  - Complex
  - Several algorithms
- Overflow?
  - 32 b operands ➝ result 64 b?
- Simpler, if only unsigned numbers
  - Just multiple additions
  - Or additions and shifts
    - E.g., : 5 * => add, shift, shift, add

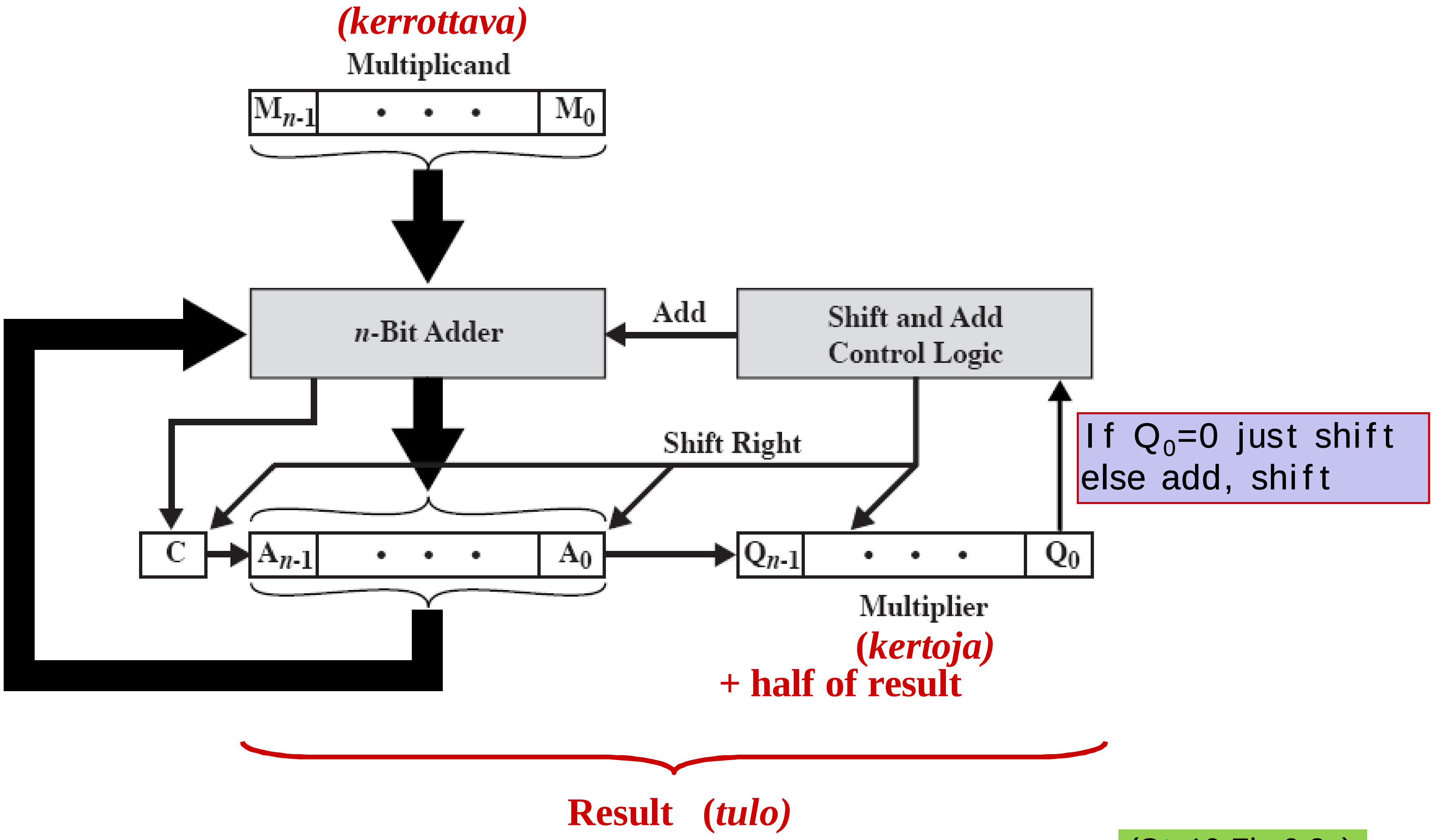


(Sta10 Fig 9.7)

```
                                 multiplier      multiplicant
Example 5*11  5=101, 11 = 1011...
101 add, shift:     add => 1011...
                  shift => 01011..
101 shift:        shift => 001011.
101 add, shift:     add => 110111.
result= 55:       shift => 0110111
```

Discussion?

Unsigned multiplication example

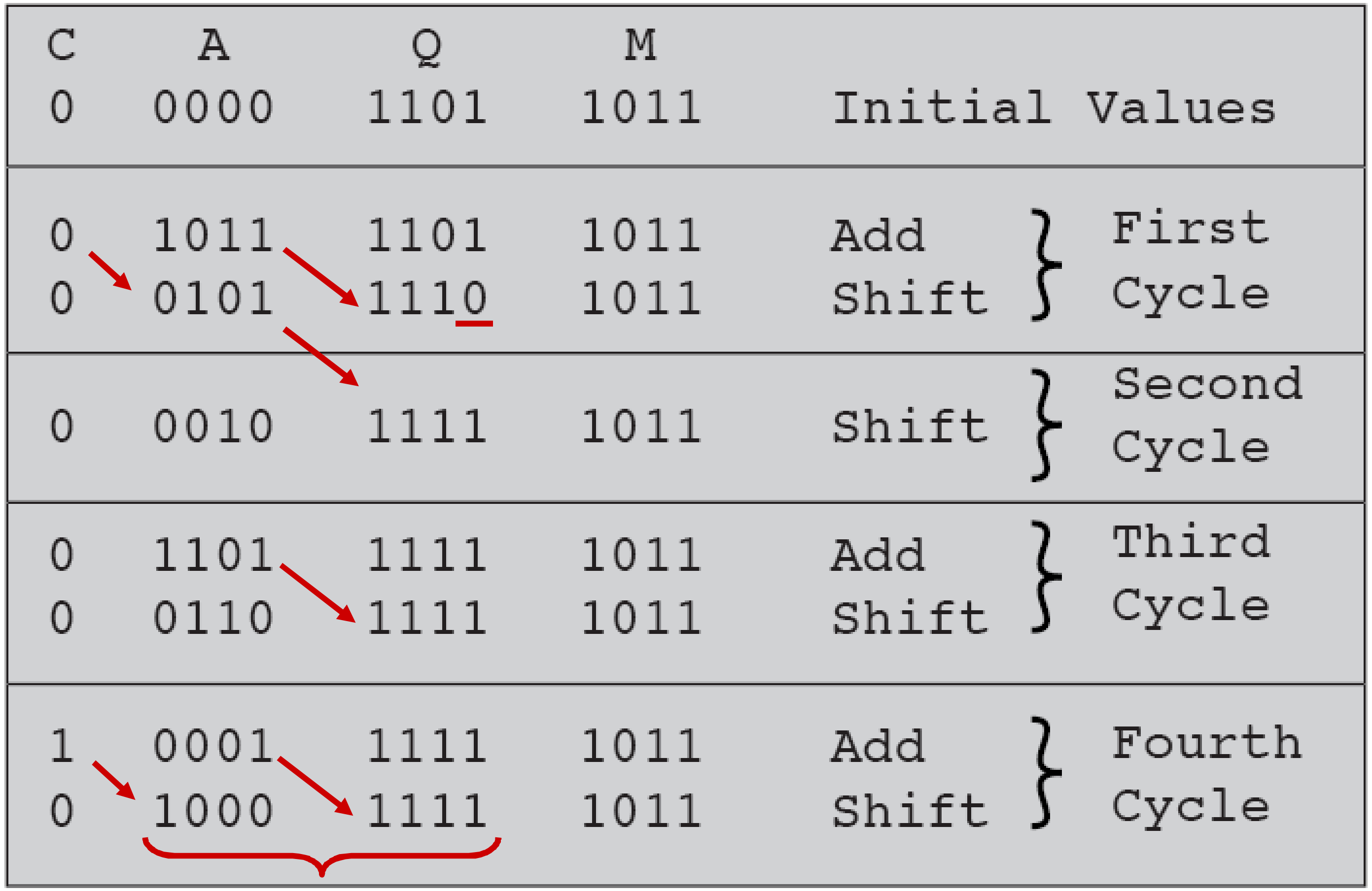


If Q_0=0 just shift
else add, shift

(Sta10 Fig 9.8a)

Unsigned multiplication

Q * M = 1101 * 1011 = 1000 1111, i.e., 13*11 = 143



C	A	Q	M		
0	0000	1101	1011	Initial Values	
0	1011	1101	1011	Add	First Cycle
0	0101	1110	1011	Shift	
0	0010	1111	1011	Shift	Second Cycle
0	1101	1111	1011	Add	Third Cycle
0	0110	1111	1011	Shift	
1	0001	1111	1011	Add	Fourth Cycle
0	1000	1111	1011	Shift	

(b) Example from Figure 9.7 (product in A, Q)

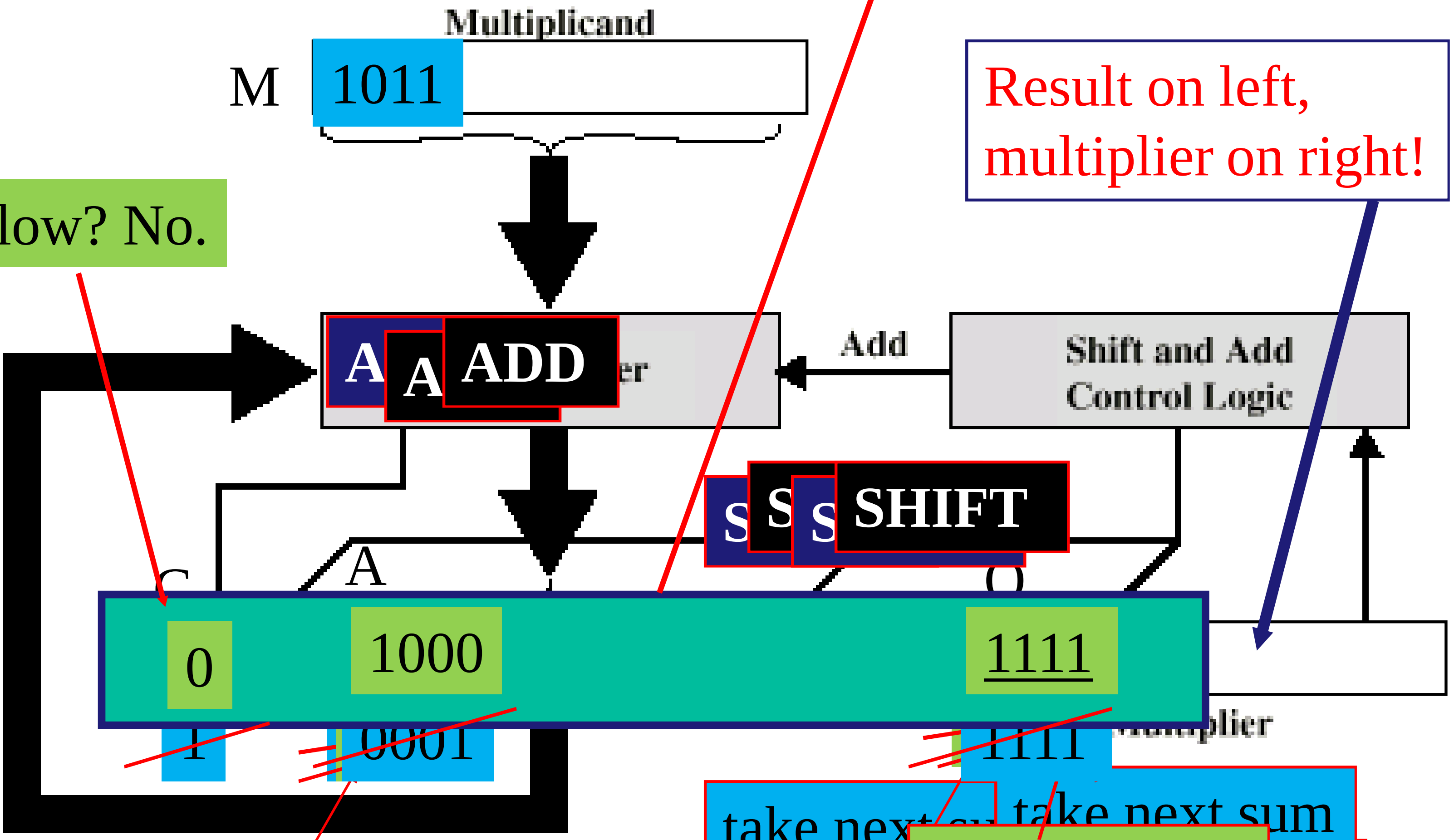
(Sta10 Fig 9.8b)

Unsigned Multiplication Example

13 * 11 = ???

= 1000 1111 = 128+8+4+2+1 = 143

(Sta10 Fig. 9.8)



Result on left, multiplier on right!

Overflow? No.

from C

result bit from A

take next su

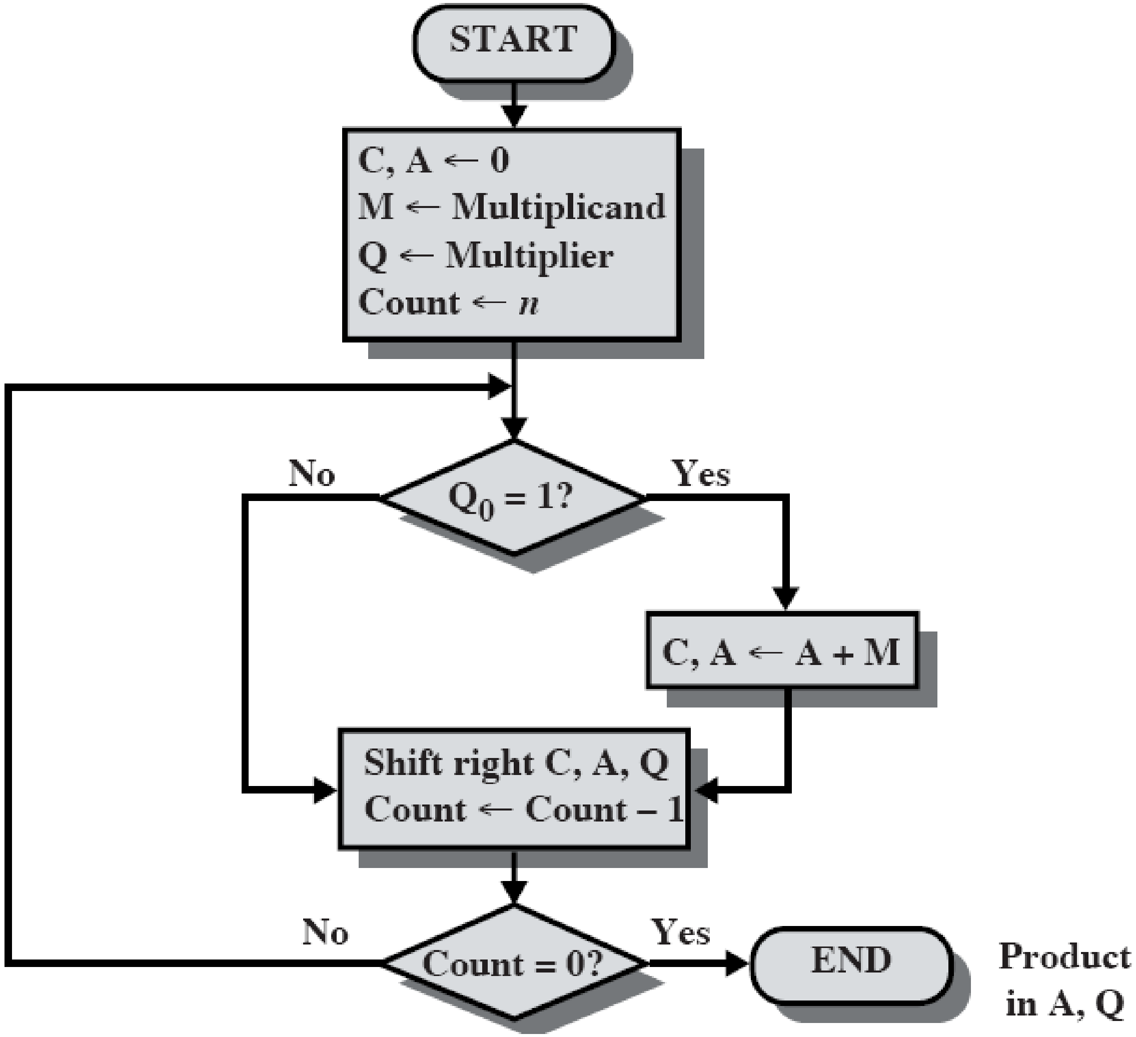
take next sum

take next sum

um

just do SHIFT

Unsigned multiplication



(Sta10 Fig 9.9)

Multiplication with negative values?

- The preceeding algorithm for unsigned numbers does NOT work for negative numbers
- Could do with unsigned numbers
  - ❶ Change operands to positive values
  - ❷ Do multiplication with positive values
  - ❸ Check signs and negate the result if needed
- This works, but there are better and faster mechanisms available

Booth’s Algorithm

- Unsigned multiplication:
  - Addition (only) for every ”1” bit in multiplier (*kertoja*)
- Booth’s algorithm (improvement)
  - Combine all adjacent 1’s in multiplier together,
  - Replace all additions by one subtraction and one addition
  - Example: decimal: 7*x = 8*x +(–x)
  - Binary: 111*x = 1000*x +(-x) =
  - add, shift, shift, shift, complement, add

    (in reality, the complement/add would be first)

5 * 7 = 0101 * 0111
= 0101 * (1000-0001)

$$\begin{array}{rr} 00101\underline{000} & 40 \\ \underline{11111011} & \underline{-5} \\ 1\underline{0}0100011 & = 35 \end{array}$$

- Works for twos complement! Also negative values!

Discussion?

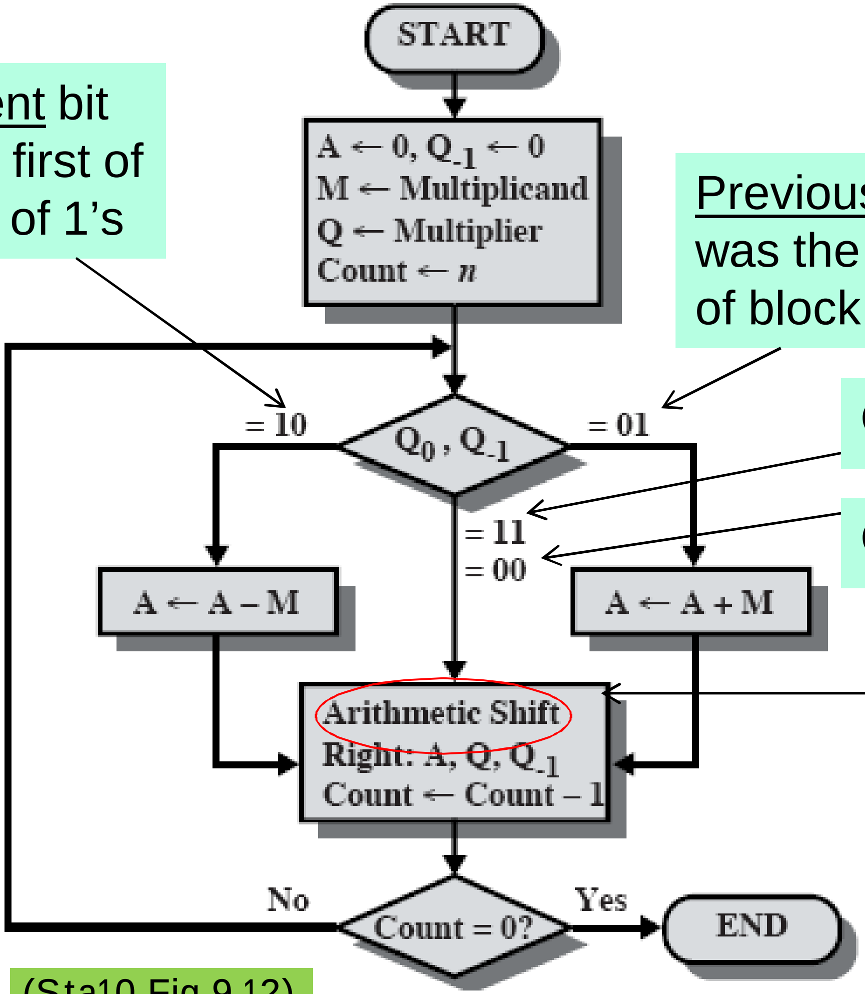
Booth's algorithm

Arithmetic Shift Right:
= fill with sign

1000 1000

1100 0100

Sign bit extending



Current bit is the first of block of 1's

Previous bit was the last of block of 1's

Continuing block of 1's

Continuing block of 0's

(Sta10 Fig 9.12)

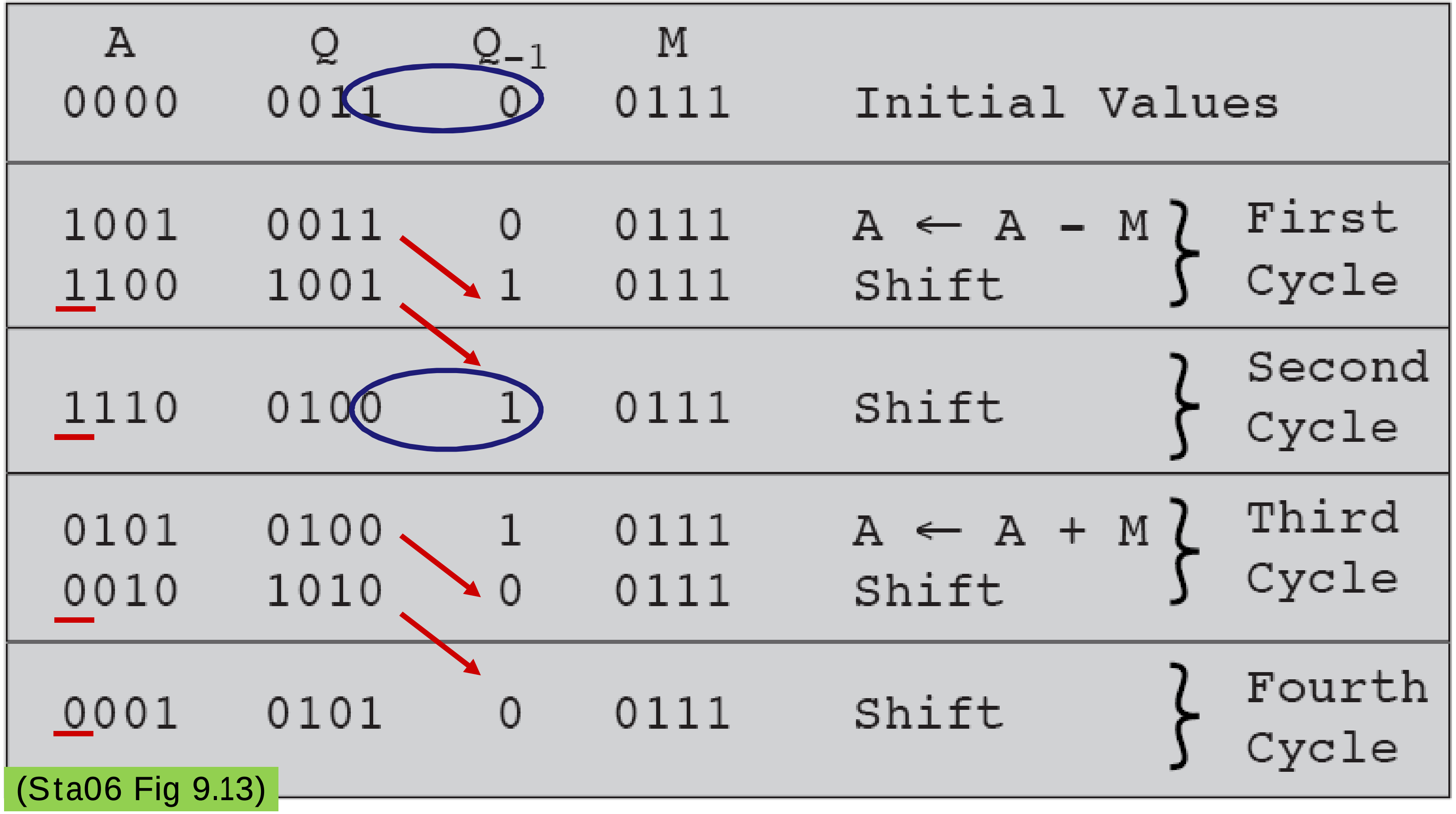
Why does it work?
$M*(01111111) = 2^7 - 1$
$M*(00011110) = 2^5 - 2^1$
$M*(01111010) = 2^7 - 2^3 + 2^2 - 2^1$

Booth’s Algorithm, example

Q * M = 0011 * 0111 = 0001 0101 eli 3*7 = 21

1-0 subtract (*vähennys*)

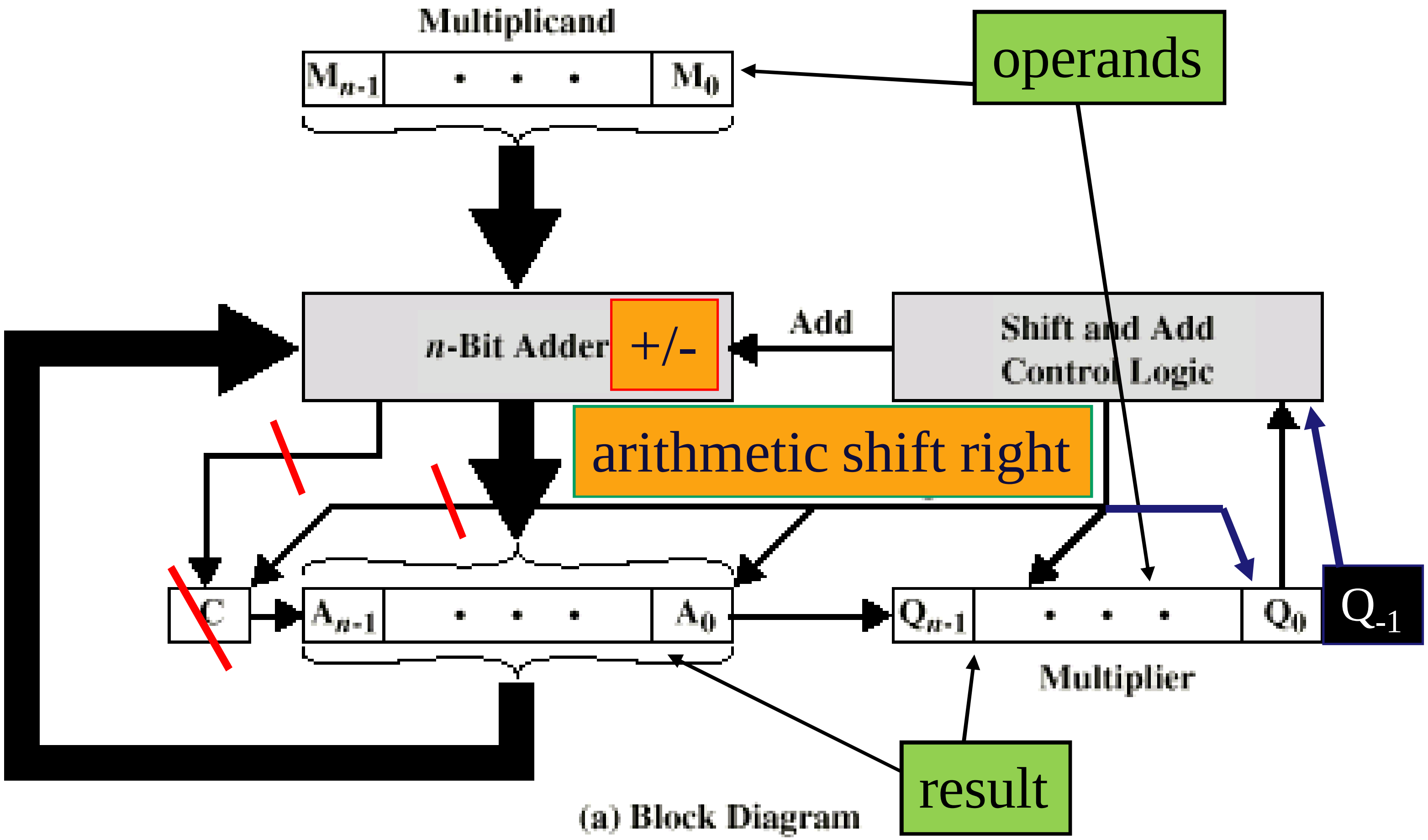
0-1 add (*lisäys*)



A	Q	Q_{-1}	M		
0000	0011	0	0111	Initial Values	
1001	0011	0	0111	A ← A − M	First Cycle
1100	1001	1	0111	Shift	
1110	0100	1	0111	Shift	Second Cycle
0101	0100	1	0111	A ← A + M	Third Cycle
0010	1010	0	0111	Shift	
0001	0101	0	0111	Shift	Fourth Cycle

(Sta06 Fig 9.13)

Booth’s Algorithm for Twos Complement Multiplication

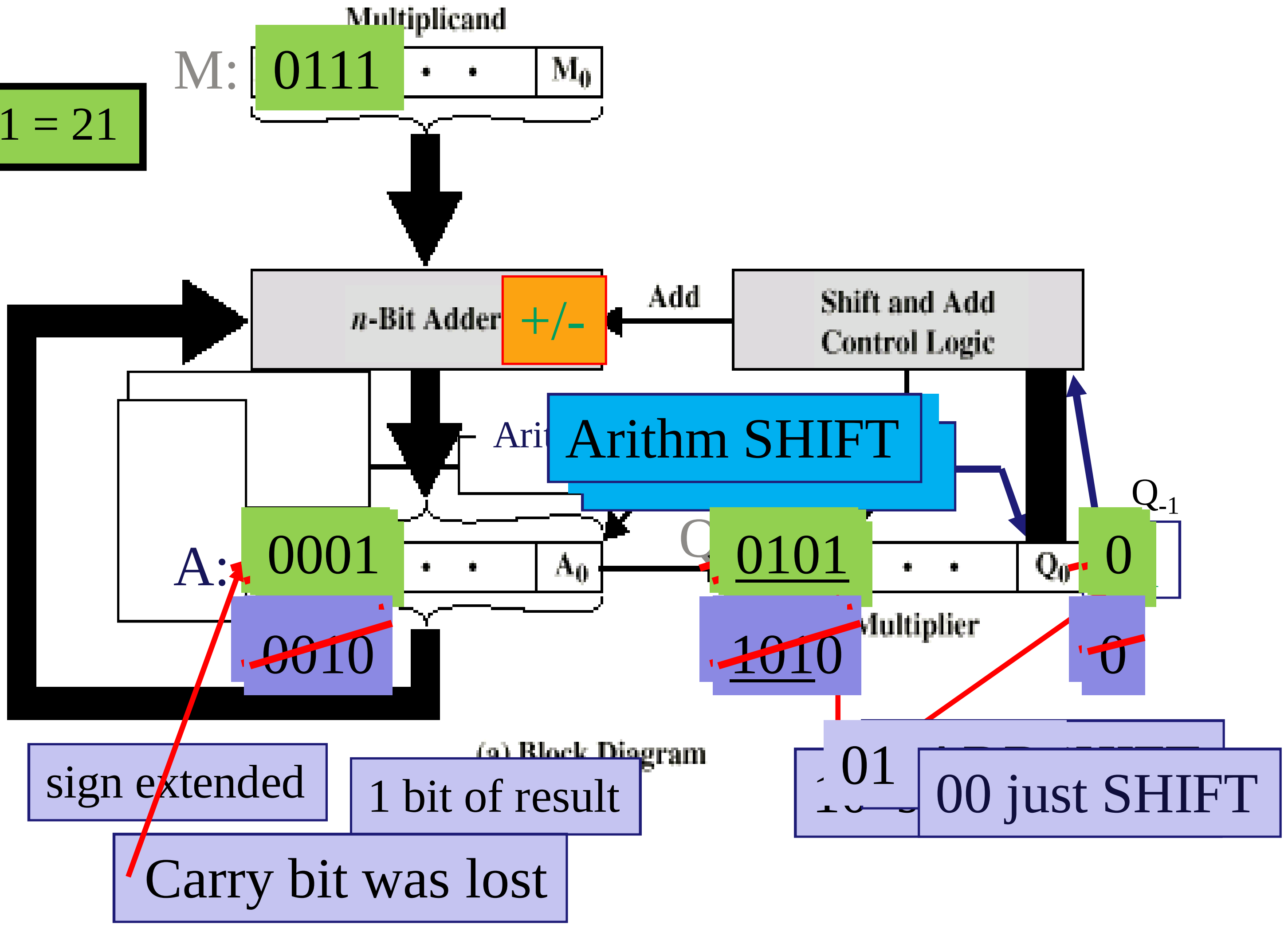


(a) Block Diagram

Booth’s Algorithm Example

7 * 3 = ?

= 0001 0101 = 21



Arithm SHIFT

sign extended

1 bit of result

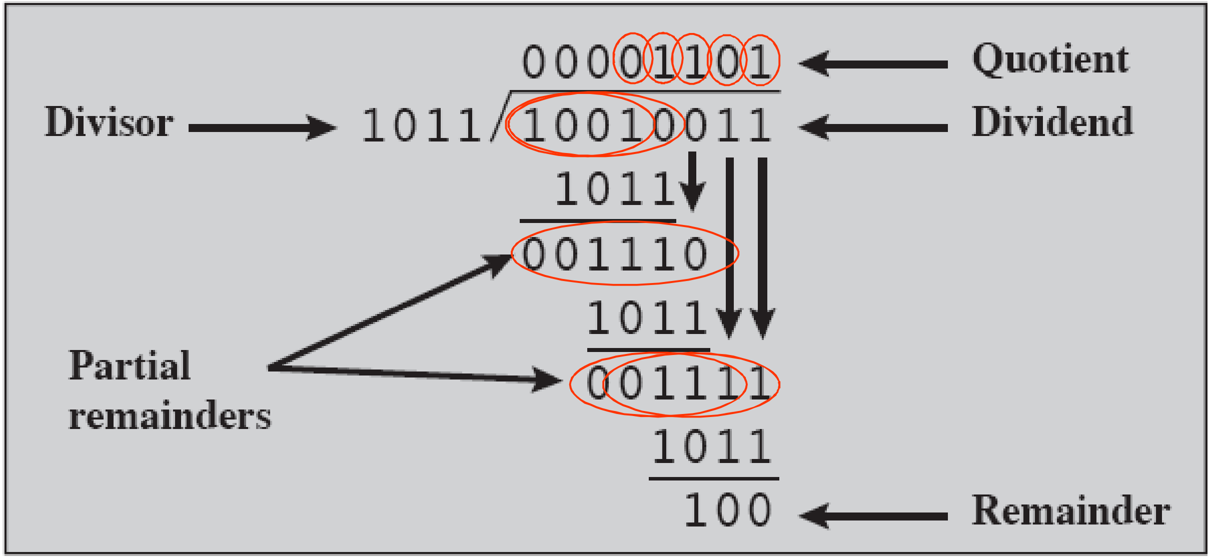
Carry bit was lost

01

00 just SHIFT

Integer division

- Like in school algorithm
  - Easy: new quotient digit always 0 or 1



(jakaja)

(osamäärä)

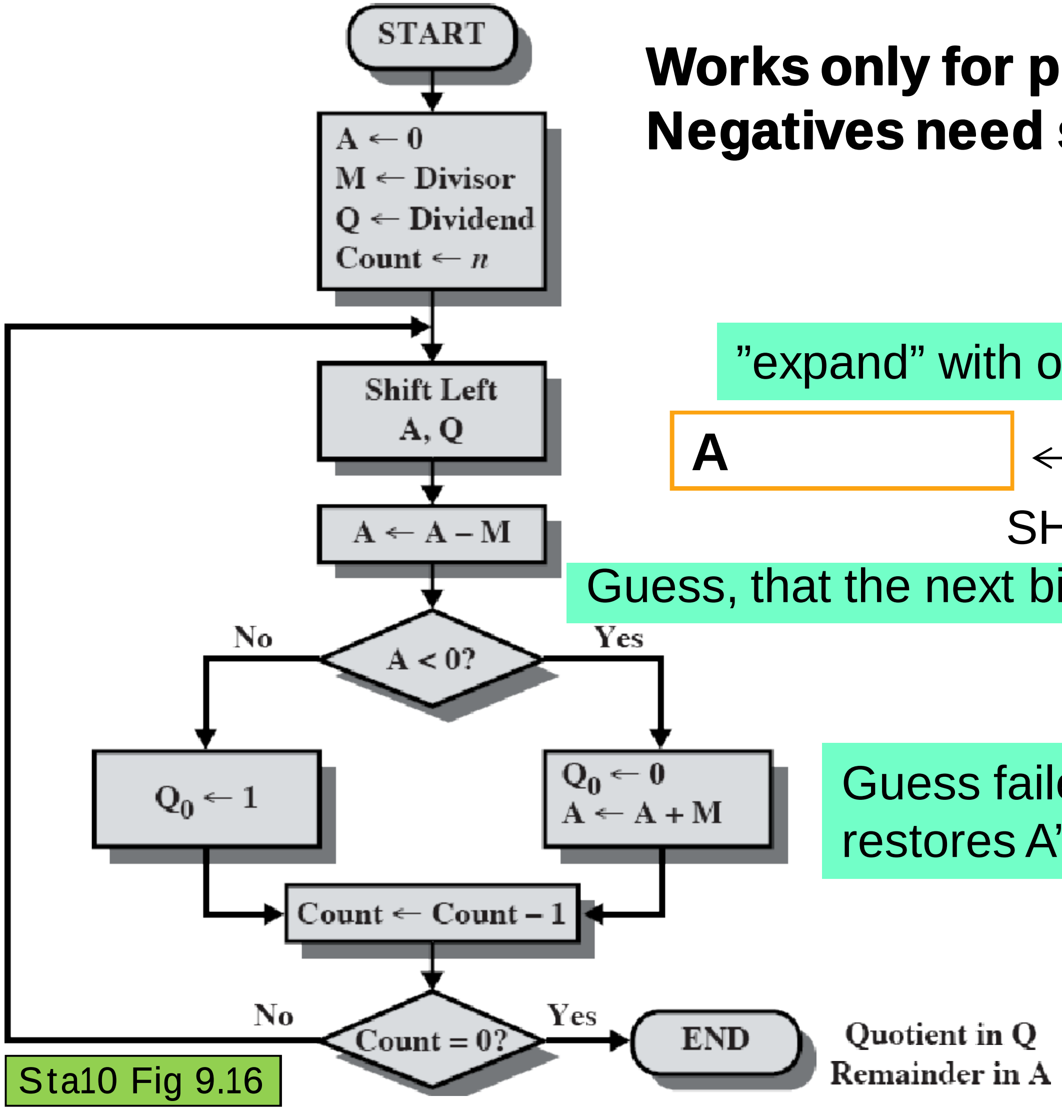
(jaettava)

(jakojäännös)

(Sta10 Fig 9.15)

- Hardware needs as in multiplication
  - Consider new digit? -- shift left (etc.)

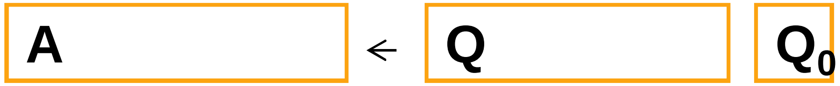
Integer division



Sta10 Fig 9.16

Works only for positive integers,
Negatives need some extra work.

”expand” with one more digit



SHL

Guess, that the next bit is 1

Guess failed,
restores A’s previous value

Example: twos complement division

- Division: 7/3 A+ Q = 7 = 0000 0111 M= 3 = 0011

A	Q	
0000	0111	initial value
0000	1110	shift left
1101		subtract M
0000	1110	restore
0001	1100	shift left
1110		subtract M
0001	1100	restore
0011	1000	shift left
0000		subtract M
0000	1001	set Q_0=1
0001	0010	shift
1110		subtract M
0001	0010	restore

Sta10 Fig 9.17

Subtract M = Add (–M)

-M = -3 = 1101

First try, if you can do the subtraction (or add if different signs).
If the sign changed, subtraction failed and A must be restored, Q_0 = 0

If subtraction succesful, Q_0 = 1

Q = quotient = 2

A = remainder = 1

Repeat as many times as Q has bits.

Floating Point Representation

- Significant digits *(Merkitsevät numerot)* and exponent *(suuruusluokka)*
- Normalized number (*Normeerattu muoto*)
  - Most significant digit is nonzero >0
  - Commonly just one digit before the radix point (*desim. pilkku*)

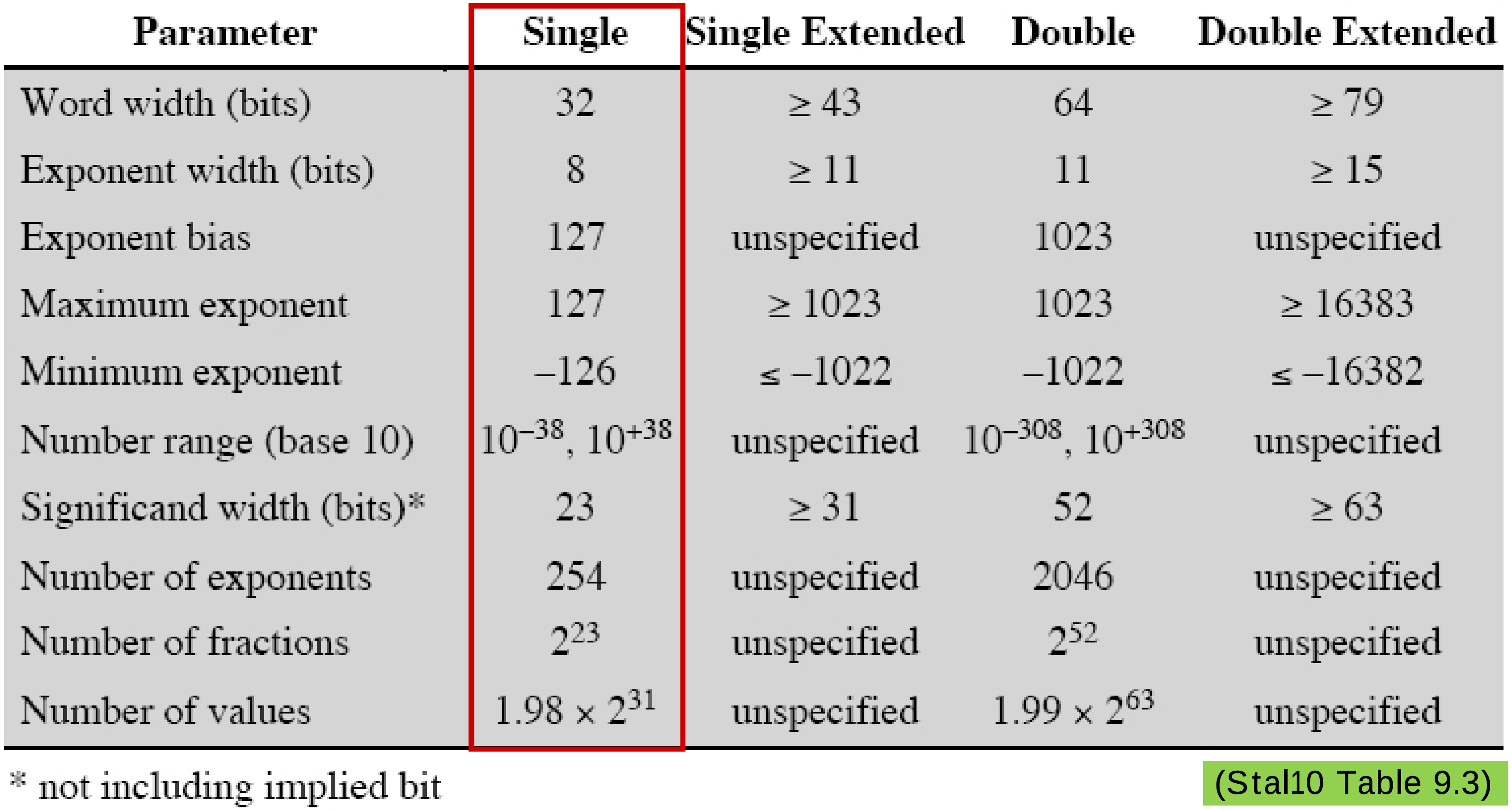
$-0.000\ 000\ 000\ 123 = -1.23 * 10^{-10}$

$0.123 = +1.23 * 10^{-1}$

$123.0 = +1.23 * 10^{2}$

$123\ 000\ 000\ 000\ 000 = +1.23 * 10^{14}$

IEEE 754 (floating point) formats



Parameter	Single	Single Extended	Double	Double Extended
Word width (bits)	32	≥ 43	64	≥ 79
Exponent width (bits)	8	≥ 11	11	≥ 15
Exponent bias	127	unspecified	1023	unspecified
Maximum exponent	127	≥ 1023	1023	≥ 16383
Minimum exponent	–126	≤ –1022	–1022	≤ –16382
Number range (base 10)	10^{-38}, 10^{+38}	unspecified	10^{-308}, 10^{+308}	unspecified
Significand width (bits)*	23	≥ 31	52	≥ 63
Number of exponents	254	unspecified	2046	unspecified
Number of fractions	2^{23}	unspecified	2^{52}	unspecified
Number of values	1.98×2^{31}	unspecified	1.99×2^{63}	unspecified

* not including implied bit

(Stal10 Table 9.3)

32-bit floating point

- 1 b sign
  - 1 = “-”, 0 = “+”
- 8 b exponent
  - Biased representation, no sign (*Ei etumerkkiä, vaan erillinen nollataso, talletus vakiolisäyksellä*)
    - Exp=5 → store 127+5, Exp=-5 → store 127-5 (bias127)
- 23 b significant (*mantissa*)
  - In normalized form the radix point is preceeded with 1, which is not stored. (hidden bit, Zuse Z3 1939)
- The binary value of the floating point representation

$$-1^{\mathbf{Sign}} * \mathbf{1.Mantissa} * 2^{\mathbf{Exponent-127}}$$

Example

$23.0 = +10111.0 * 2^0 = +1.0111 * 2^4 =?$

127+4=131

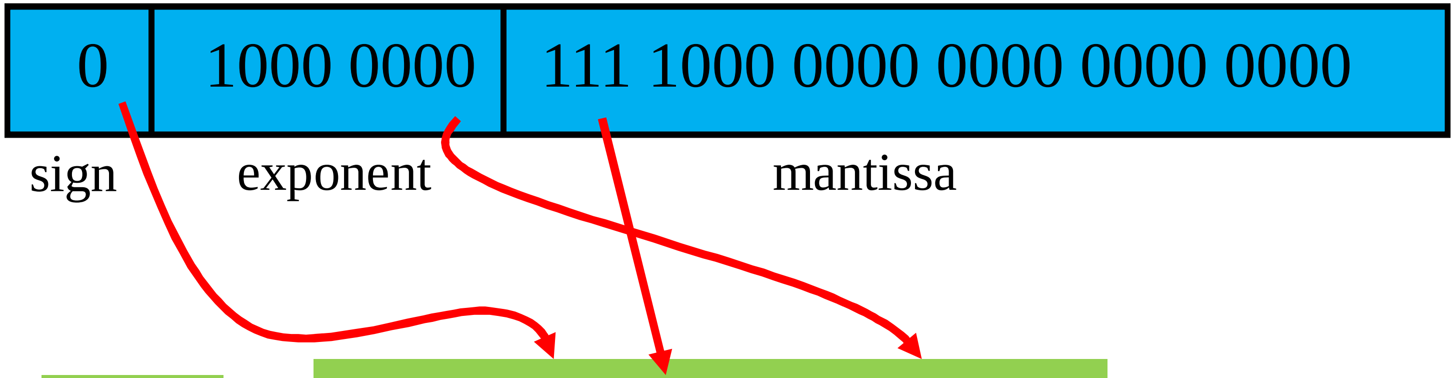
0	1000 0011	011 1000 0000 0000 0000 0000
sign	exponent	mantissa

$1.0 = +1.0000 * 2^0 = ?$

0+127 = 127

0	0111 1111	000 0000 0000 0000 0000 0000
sign	exponent	mantissa

Example



0	1000 0000	111 1000 0000 0000 0000 0000
sign	exponent	mantissa

X = ?

$X = (-1)^0 * 1.1111 * 2^{(128-127)}$

$= 1.1111_2 * 2$

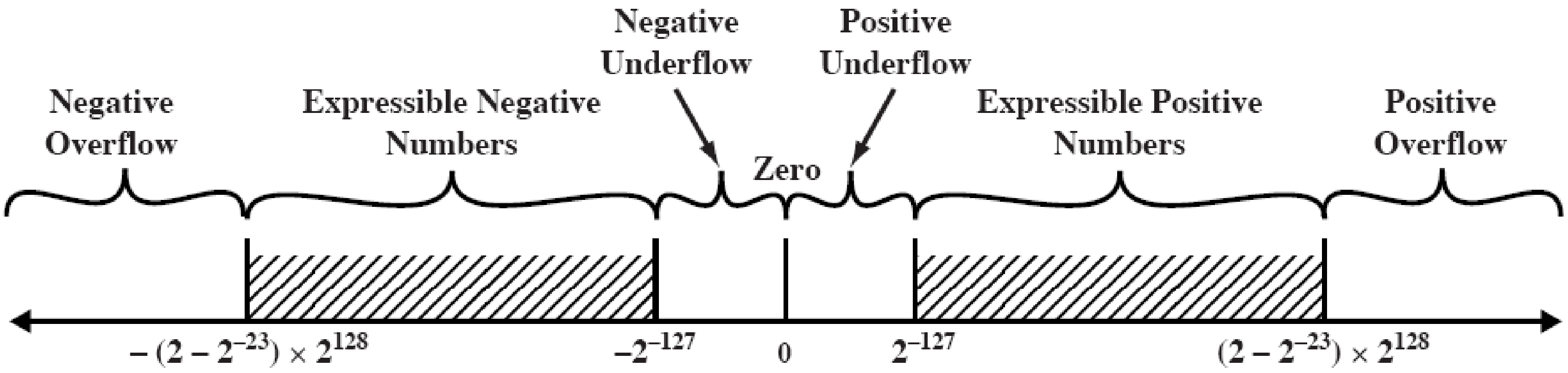
$= (1+ 1/2 + 1/4 + 1/8 + 1/16) * 2$

$= (1 + 0.5 + 0.25 + 0.125+0.0625) * 2$

$= 1.9375 * 2$

$\boxed{= 3.875}$

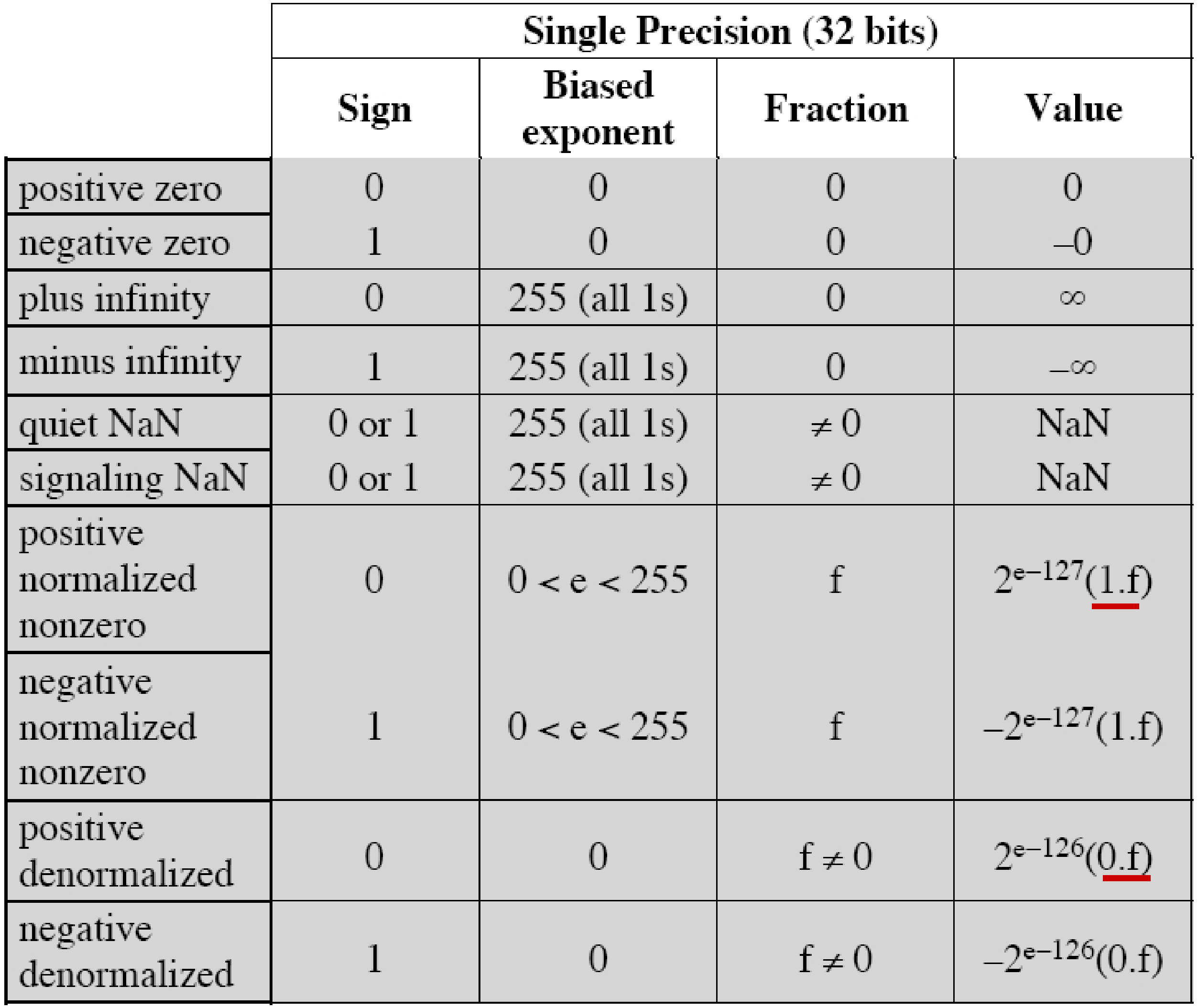
Accuracy (*tarkkuus*) (32b)



- Value range (*arvoalue)*
  - 8 b exponent→ 2^{-126} ... 2^{127} ~ -10^{-38} ... 10^{38}
- Not exact value
  - 24 b mantissa → 2^{24} ~ 1.7 * 10^{-7} ~ 6 decimals
- Balancing between range and precision

Numerical errors: Patriot Missile (1991), Ariane 5 (1996)
http://ta.twi.tudelft.nl/nw/users/vuik/wi211/disasters.html

Interpretation of IEEE 754 Floating-Point Numbers



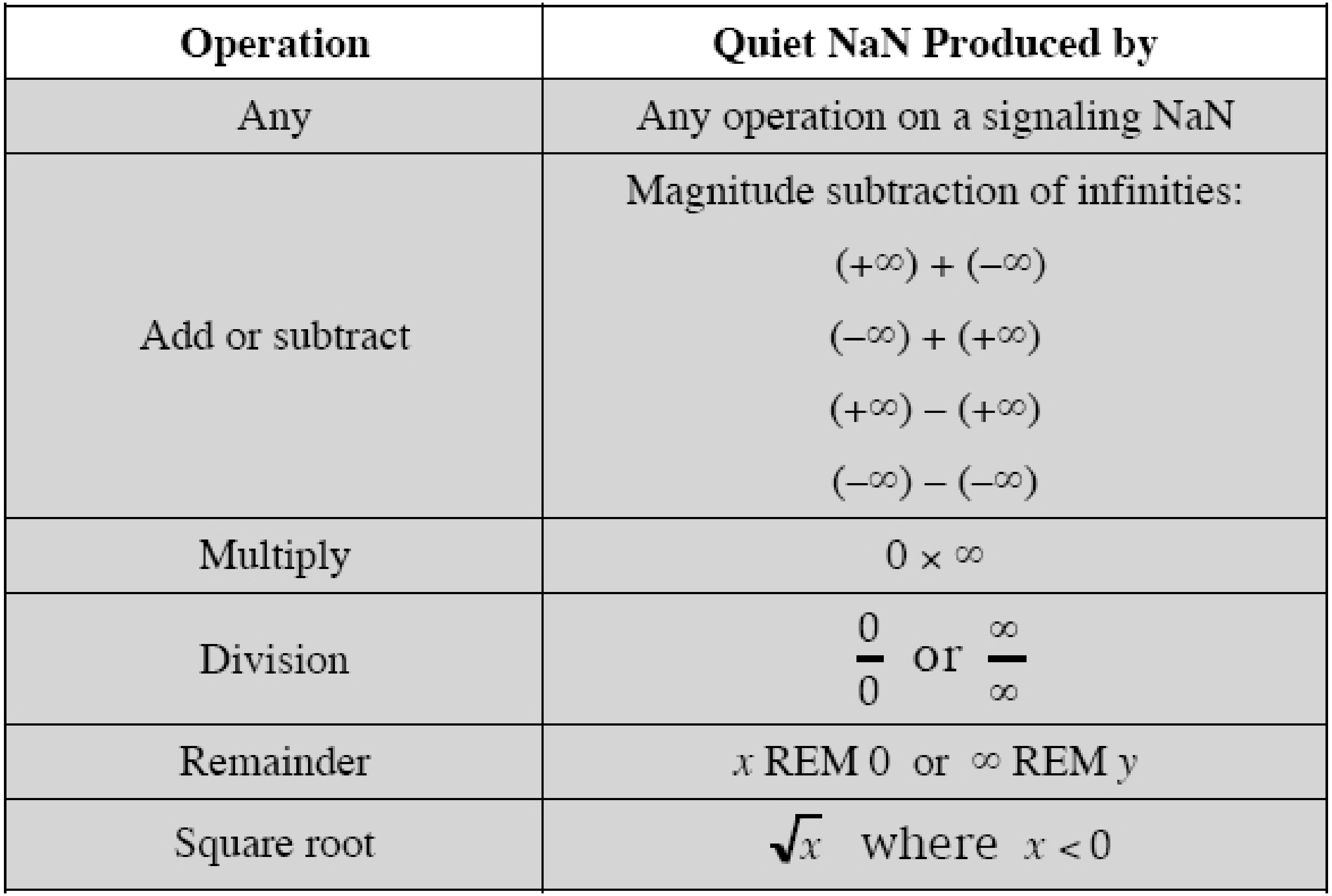
	Single Precision (32 bits)			
	Sign	Biased exponent	Fraction	Value
positive zero	0	0	0	0
negative zero	1	0	0	–0
plus infinity	0	255 (all 1s)	0	∞
minus infinity	1	255 (all 1s)	0	$-\infty$
quiet NaN	0 or 1	255 (all 1s)	$\neq 0$	NaN
signaling NaN	0 or 1	255 (all 1s)	$\neq 0$	NaN
positive normalized nonzero	0	$0 < e < 255$	f	$2^{e-127}(\underline{1.f})$
negative normalized nonzero	1	$0 < e < 255$	f	$-2^{e-127}(1.f)$
positive denormalized	0	0	$f \neq 0$	$2^{e-126}(\underline{0.f})$
negative denormalized	1	0	$f \neq 0$	$-2^{e-126}(0.f)$

(Sta10 Table 9.4)

Not a Number

Double Precision similarly

NaN: Not a Number



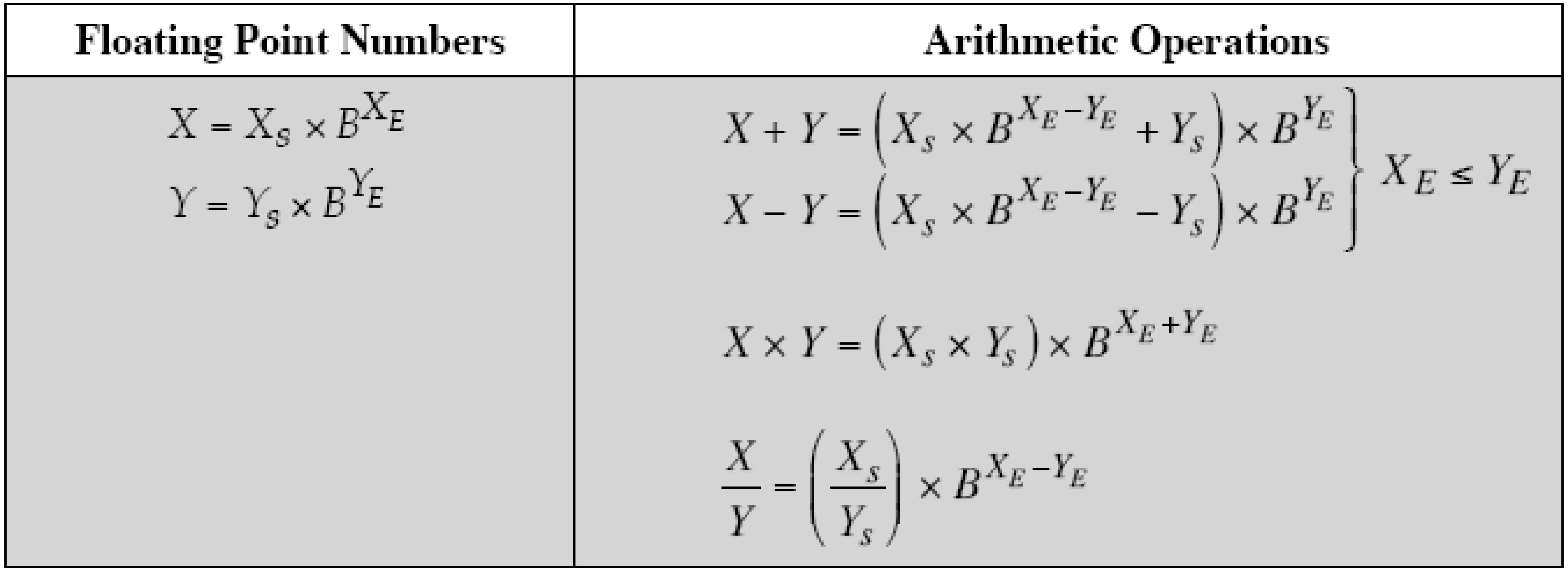
Operation	**Quiet NaN Produced by**
Any	Any operation on a signaling NaN
Add or subtract	Magnitude subtraction of infinities: $(+\infty) + (-\infty)$ $(-\infty) + (+\infty)$ $(+\infty) - (+\infty)$ $(-\infty) - (-\infty)$
Multiply	$0 \times \infty$
Division	$\frac{0}{0}$ or $\frac{\infty}{\infty}$
Remainder	x REM 0 or ∞ REM y
Square root	$\sqrt{x}$ where $x < 0$

(Sta10 Table 9.6)

Floating Point Arithmetics

- Calculations need wide registers
  - Guard bits - pad right end of significand
  - More bits for the significand (mantissa)
  - Using denormalized formats
- Addition and subtraction
  - More complex than multiplication
  - Operands must have same exponent
    - Denormalize the smaller operand (alignment!)
      - Loss of digits (less precise and missing information)
  - Result (must) be normalised
- Multiplication and division
  - Significand and exponent handled separately

Floating Point Arithmetics



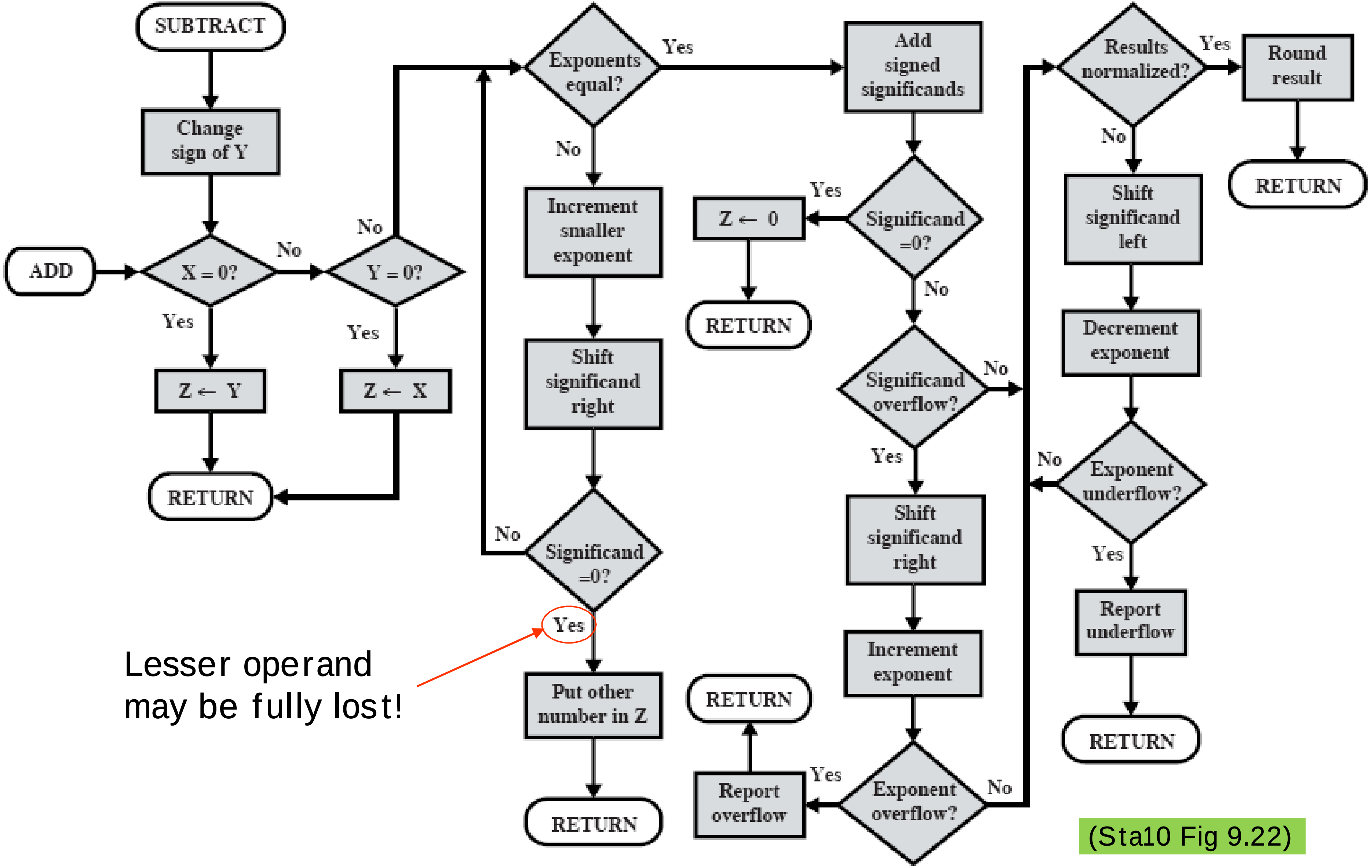
Floating Point Numbers	Arithmetic Operations
$X = X_S \times B^{X_E}$ $Y = Y_S \times B^{Y_E}$	$\left.\begin{array}{l} X + Y = \left(X_S \times B^{X_E - Y_E} + Y_S\right) \times B^{Y_E} \\ X - Y = \left(X_S \times B^{X_E - Y_E} - Y_S\right) \times B^{Y_E} \end{array}\right\} X_E \leq Y_E$ $X \times Y = \left(X_S \times Y_S\right) \times B^{X_E + Y_E}$ $\frac{X}{Y} = \left(\frac{X_S}{Y_S}\right) \times B^{X_E - Y_E}$

(Sta10 Table 9.5)

$$X = 0.3 \times 10^2 = 30$$
$$Y = 0.2 \times 10^3 = 200$$

$$X + Y = (0.3 \times 10^{2-3} + 0.2) \times 10^3 = 0.23 \times 10^3 = 230$$
$$X - Y = (0.3 \times 10^{2-3} - 0.2) \times 10^3 = (-0.17) \times 10^3 = -170$$
$$X \times Y = (0.3 \times 0.2) \times 10^{2+3} = 0.06 \times 10^5 = 6000$$
$$X \div Y = (0.3 \div 0.2) \times 10^{2-3} = 1.5 \times 10^{-1} = 0.15$$

Floating Point Addition and Subtraction



Lesser operand may be fully lost!

(Sta10 Fig 9.22)

Floating Point Special Cases

- Exponent overflow (*eksponentin ylivuoto*)
  - Very large number (above max)
  - Value ∞ or -∞ , alternatively cause exception

Programmable option

- Exponent underflow (*eksponentin alivuoto*)
  - Very small number (below min)
  - Value 0 (or cause exception)

Programmable option

- Significand overflow (*mantissan ylivuoto*)
  - Normalise!

Fix it!

- Significand underflow (*mantissan alivuoto*)
  - Denormalizing may lose the significand accuracy
  - All significant bits lost?

Ooops, lost some or all data!

Floating Point Rounding (pyöristys)

- Example
  - Value has four decimals
  - Represent it using only 3 decimals

3.1236, -4.5678

  - Normal rounding rule

    round to nearest value

3.124, -4.568

  - Always towards ∞ (*ylöspäin*)

3.124, -4.567

  - Always towards -∞ (*alaspäin*)

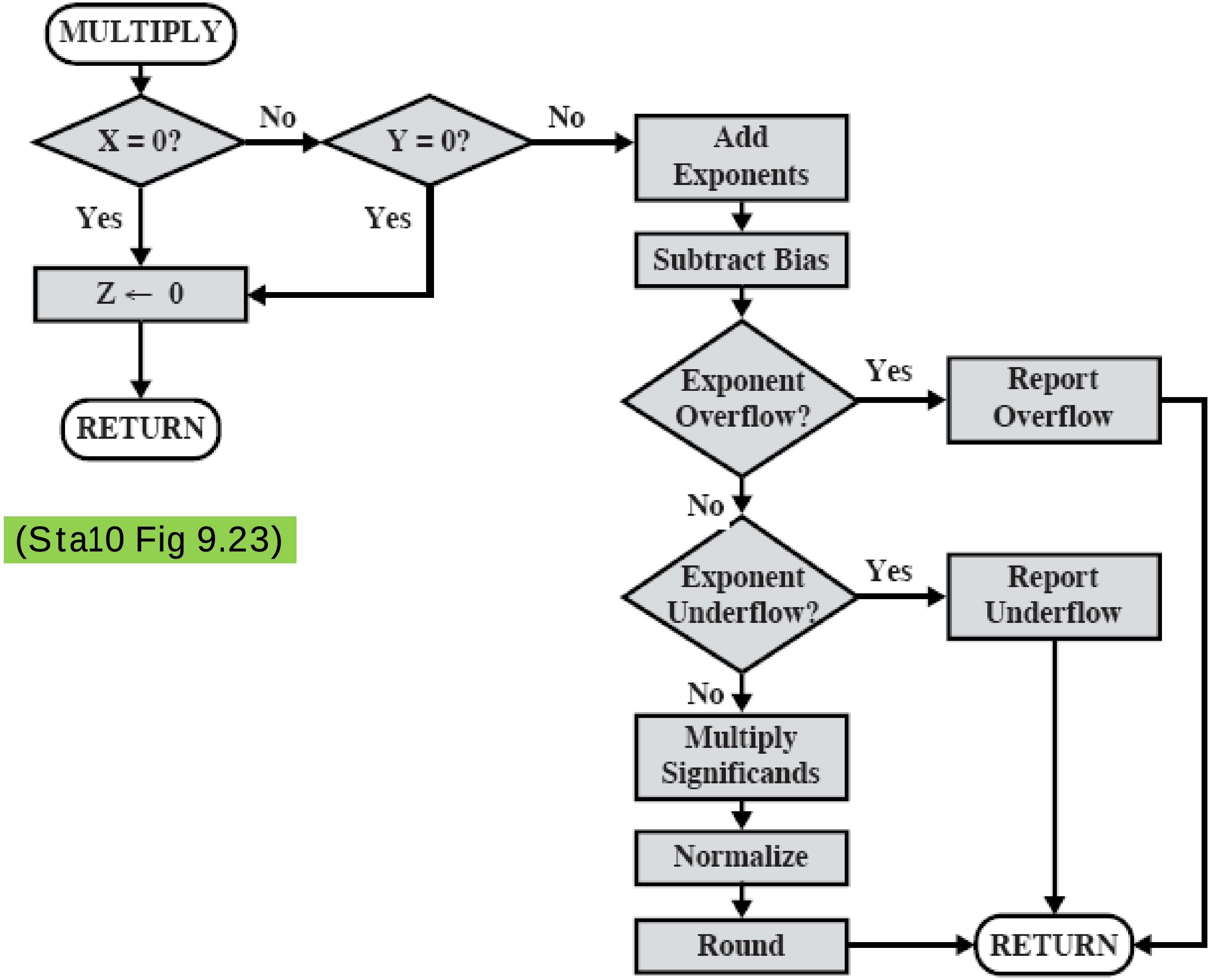
3.123, -4.568

  - Always towards 0

3.123, -4.567

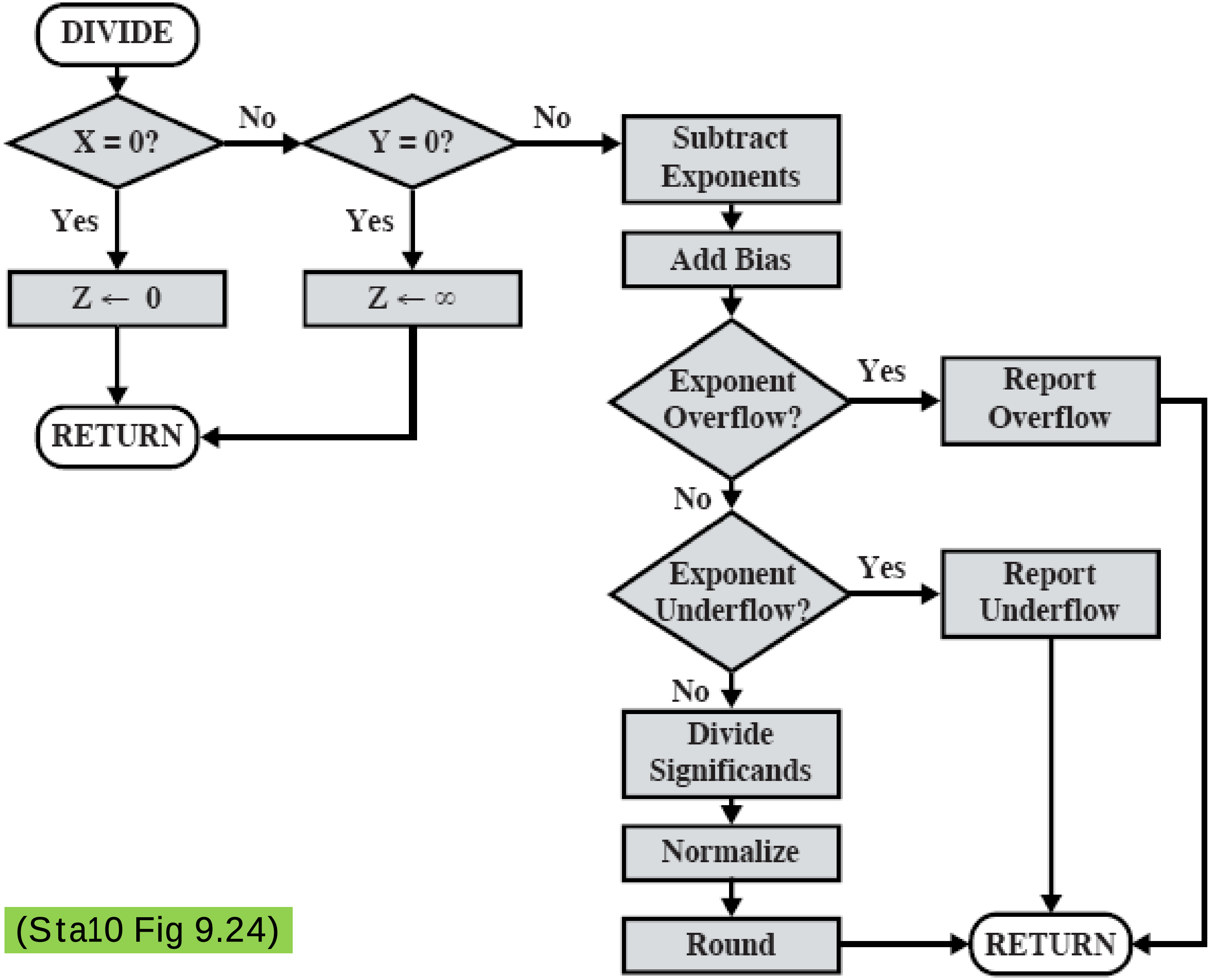
- Intel Itanium (e.g.) supports all of these alternatives

Floating Point Multiplication



(Sta10 Fig 9.23)

Floating Point Division



(Sta10 Fig 9.24)

Computer Arithmetics Summary

- Integer ops
  - 2’s complement representation
  - Negation, addition, subtraction, multiplication, division
  - Booth algorithm for multiplication
- Floating point ops
  - Complete IEEE format
    - +/- ∞, NaN, denormalized numbers, double
  - Addition, subtraction, multiplication, division
  - Overflows, underflows
  - Rounding
  - Accuracy – beware of early subtractions!

$$(1.0666668-1.0666666) * 1.23456 \neq$$
$$1.0666668*1.23456 - 1.0666666*1.23456$$

Try it out with
32-bit IEEE?

Review Questions / Kertauskysymyksiä

- Why we use twos complement?
- How does twos complement “expand” to a large number of bits (8b →16 b)?
- Format of single-precision floating point number?
- When does underflow happen?
- When can you lose accuracy?